

STRICHARTZ ESTIMATES AND THE NONLINEAR SCHRÖDINGER EQUATION FOR THE SUBLAPLACIAN ON COMPLEX SPHERES

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ABSTRACT. We study the nonlinear Schrödinger equation associated with the sublaplacian \mathcal{L} on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ equipped with its natural CR structure.

We first prove Strichartz estimates with fractional loss of derivatives for the solutions of the free Schrödinger equation and we then deduce some local in time well-posedness results. Our results are stated in terms of certain Sobolev-type spaces, that measure the regularity of functions on S^{2n+1} differently according to their spectral localization. Stronger conclusions are obtained for particular classes of solutions, corresponding to initial data whose spectrum is contained in a proper cone of \mathbb{N}^2 .

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In this paper we study the nonlinear Schrödinger equation associated with the sublaplacian \mathcal{L} on the complex sphere S^{2n+1} in \mathbb{C}^{n+1} , $n \geq 1$

$$\begin{cases} i\partial_t v + \mathcal{L}v = F(v) \\ v(0, z) = v_0, \end{cases} \quad (1.1)$$

where $v_0 \in L^2(S^{2n+1})$, F is a nonlinear polynomial and \mathcal{L} denotes the sublaplacian, that is, the operator defined by

$$\mathcal{L} := - \sum_{1 \leq j < k \leq n+1} M_{jk} \overline{M}_{jk} + \overline{M}_{jk} M_{jk}, \quad (1.2)$$

with $M_{jk} := \overline{z}_j \partial_{z_k} - \overline{z}_k \partial_{z_j}$. The operator \mathcal{L} is a densely defined, self-adjoint, positive, and subelliptic operator on S^{2n+1} [Ge] and it coincides with the real part of the Kohn-Laplacian acting on functions [Lee]; see also [MPR]. The sublaplacian \mathcal{L} may be considered as the subriemannian analogue of the Laplace–Beltrami operator on a Riemannian manifold, see e.g. [JeLee].

Our first result is a Strichartz estimate for the solution v of the free Schrödinger equation

$$\begin{cases} i\partial_t v + \mathcal{L}v = 0 \\ v(0, z) = v_0. \end{cases} \quad (1.3)$$

Strichartz estimates are a family of space-time bounds on solutions of (1.1), which provide a useful tool to control the norm of the solutions and to prove local well-posedness results. In this case, we bound the $L_t^p L_x^q$ norm of v by means of a suitable mixed Sobolev norm, denoted by $\|v_0\|_{\mathcal{X}^{(r,s)}}$, of the initial datum.

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In order to describe the mixed Sobolev spaces $\mathcal{X}^{(r,s)}$, we start from the classical decomposition of the space of square integrable functions on S^{2n+1}

$$L^2(S^{2n+1}) = \bigoplus_{\ell, \ell'=0}^{\infty} \mathcal{H}^{\ell, \ell'},$$

$\mathcal{H}^{\ell, \ell'}$ being the space of complex spherical harmonics of bidegree (ℓ, ℓ') , [ViK, Ch. 11].

This decomposition is the joint spectral decomposition of \mathcal{L} and the Laplace–Beltrami operator Δ on S^{2n+1} , since the subspaces $\mathcal{H}^{\ell, \ell'}$ are eigenspaces both for Δ with eigenvalue $\mu_{\ell, \ell'} = (\ell + \ell')(\ell + \ell' + n)$, and for \mathcal{L} with eigenvalue $\lambda_{\ell, \ell'} = 2\ell\ell' + n(\ell + \ell')$.

Now fix $M \geq 1$. We observe that if (ℓ, ℓ') belongs to the proper angular sector \mathcal{V} in \mathbf{N}^2 given by

$$\mathcal{V} = \{(\ell, \ell') \in \mathbf{N}^2 : \ell/M \leq \ell' \leq M\ell\}, \quad (1.4)$$

then $\mu_{\ell, \ell'}$ is controlled above and below by a constant times $\lambda_{\ell, \ell'}$, while if (ℓ, ℓ') belongs to the complementary region, that is, if $0 \leq \ell' < \ell/M$ or $M\ell < \ell'$, then $\mu_{\ell, \ell'}$ grows as ℓ^2 , while $\lambda_{\ell, \ell'}$ varies between ℓ and ℓ^2 .

Hence, we are led to introduce appropriate Sobolev-type spaces that measure the regularity of functions differently according to their spectral localization. For $r \geq 0$ we denote by $W^r(S^{2n+1})$ the standard non-isotropic Sobolev space, for instance defined as the image of $L^2(S^{2n+1})$ under $(I + \mathcal{L})^{-r/2}$.

We now define $\mathcal{X}^{(r,s)}(S^{2n+1})$ as the space of all functions $u \in L^2(S^{2n+1})$, spectrally decomposed as $u = \sum_{\ell, \ell'=0}^{\infty} h_{\ell, \ell'}$, $h_{\ell, \ell'} \in \mathcal{H}^{\ell, \ell'}$, such that

$$\sum_{\ell/M < \ell' < M\ell} h_{\ell, \ell'} \in W^r(S^{2n+1}),$$

while the complementary sums

$$\sum_{\ell' \leq \ell/M} h_{\ell, \ell'}, \quad \sum_{\ell' \geq M\ell} h_{\ell, \ell'} \in W^s(S^{2n+1}).$$

The Strichartz estimates that we are able to prove for solutions of (1.3) are expressed in terms of $\mathcal{X}^{(r,s)}$ -norms.

Theorem 1.1. *Let S^{2n+1} denote the unit complex sphere in \mathbf{C}^{n+1} and let \mathcal{L} be the sublaplacian, defined by (1.2). Let $p \geq 2$, $q < +\infty$ satisfy the admissibility condition*

$$\frac{2}{p} + \frac{Q}{q} = \frac{Q}{2}. \quad (1.5)$$

Define

$$s_n := \begin{cases} 2[1 - 1/(n+1)], & \text{if } n > 1 \\ 4/3, & \text{if } n = 1. \end{cases} \quad (1.6)$$

Then, if I is any finite time interval and $s \geq s_1$ or $s > s_n$ for $n > 1$, there exists a constant $C = C(s, I) > 0$ such that any solution v of (1.1) satisfies the estimate

$$\|v\|_{L^p(I, L^q(S^{2n+1}))} \leq C \|v_0\|_{\mathcal{X}^{(s/p, 2/p)}(S^{2n+1})}. \quad (1.7)$$

In order to prove the Strichartz estimates we need some kind of dispersive estimates. In Theorem 3.1 we prove spectrally localized dispersive estimates in terms of the mixed Sobolev spaces $\mathcal{X}^{(r,s)}$ and we need to take into account the oscillation of the infinite sum that defines the integral kernel of the solution operator and that depends on the two indices ℓ and ℓ' .

It may be seen as a natural fact that we need to consider a two-parameter scale for the Sobolev spaces, given the fact that the kernel depends on two indices.

The proof of Theorem 3.1 is quite delicate and occupies a good portion of the present paper.

In the final part of the paper, we apply Strichartz estimate (1.7) to prove local well-posedness for the nonlinear Schrödinger equation (1.1), with F is a nonlinear polynomial of degree β , with $F(0) = 0$. According to a classical approach, we replace the nonlinear equation in (1.1) by means of an integral equation, which may be solved by the iteration method on a suitable function space \mathcal{F} . If the Sobolev index r is sufficiently large, we can obtain well-posedness on $\mathcal{F} = \mathcal{C}([-T, T], W^r(S^{2n+1}))$ for some small $T > 0$, but if the Sobolev index r is small, the iteration method fails on this space and we need to restrict \mathcal{F} .

In our approach, the Sobolev regularity index r depends on the spectral localization of the initial datum. In particular, if the initial datum v_0 is spectrally localized in a proper cone \mathcal{V} and $v_0 \in W^r(S^{2n+1})$, for some $r > \frac{Q}{2} - \frac{1}{\max(\beta-1, 2)}$, then there exist $T > 0$ and a unique solution v of (1.1), such that

$$v \in \mathcal{C}([-T, T], W^r(S^{2n+1})) \cap L^p([-T, T], L^\infty(S^{2n+1})),$$

see Proposition 6.2.

While there exists a vast literature on the nonlinear Schrödinger equation for the Laplace–Beltrami operator on Riemannian manifolds (see the comments below), little is known in the case of the sublaplacian on CR manifolds, even in the case of the Heisenberg group \mathbf{H}_n . We recall that the Heisenberg group \mathbf{H}_n is biholomorphically equivalent to the unit sphere S^{2n+1} with the north pole removed, via the Cayley transform.

Our Theorem 1.1 should be compared with results concerning the Schrödinger equation on the Heisenberg group \mathbf{H}_n . H. Bahouri, P. Gérard and C.-J. Xu in the seminal paper [BaGX] prove that no global in time dispersive estimate may hold for solutions of the Schrödinger equation on \mathbf{H}_n ; see also the more recent work by Gérard and S. Grellier [GG1, GG2]. However, the same lack of dispersion occurs on S^{2n+1} , and, in addition, no local in time dispersive estimate can hold, as we shall observe in Section 3. Nonetheless, we are able to prove local Strichartz estimates for the solutions of (1.1), by substituting the dispersive estimate for the Schrödinger propagator $e^{it\mathcal{L}}$ by a family of dispersive estimates for the frequency localized operator $e^{it\mathcal{L}}\varphi(h^2\mathcal{L})$. This idea originally appeared in [BaCh] and [Tat], and has been successfully applied in the work of Burk, Gérard and Tzvetkov [BuGT1, BuGT2]. One might wonder if the same technique could apply to the Heisenberg framework and this topic will be the object of further investigation.

To the best of our knowledge, the results in the present paper are the first ones concerning the local well-posedness for the Schrödinger equation for a sublaplacian.

It is interesting to compare our results with the known ones in the Riemannian framework, where Strichartz estimates have been extensively studied. Consider a Riemannian manifold (M, g) of dimension d and the Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_g u = 0 \\ u(0, x) = u_0, \end{cases} \quad (1.8)$$

where Δ_g denotes the Laplace–Beltrami operator on (M, g) . Then Strichartz estimates of solutions of (1.8) are usually of the form

$$\|u\|_{L^p([-T, T], L^q(M))} \leq C \|u_0\|_{H^s(M)} \quad (1.9)$$

where p, q, d satisfy the scale-invariance condition

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}. \quad (1.10)$$

Here and in what follows, we denote by H^s the classical Sobolev space on M , which may be defined as the image of $L^2(M)$ under $(I + \Delta_g)^{-s/2}$. The key ingredient to prove (1.9) is given by some dispersive estimates, that is, estimates of the L^∞ norm of solutions of (1.8) at a fixed time t .

When $M = \mathbf{R}^n$, the theory is basically well-established and one can choose $s = 0$ and $T = \infty$ in (1.9), thanks to the essential contributions by Strichartz, Ginibre and Velo, and Keel and Tao [Str, GiV, KT].

When M is a generic Riemannian manifold, the situation is more involved and the geometry, as it is well known, plays an essential rôle. In general, non compact manifolds enjoy better decay properties, so that stronger well-posedness results may be proved in that framework. On compact manifolds the dispersion is generally weaker, since, in particular, no global in time dispersive estimate can hold. Nonetheless, Burq, Gérard and Tzvetkov, generalizing the earlier work of J. Bourgain on tori [Bou1, Bou2], proved an estimate like (1.9) on any compact and boundaryless manifold M , with $s = 1/p$, with a loss of derivatives with respect to the flat Euclidean case, but again with a gain of $1/p$ derivatives in comparison to the bounds indicated by Sobolev embeddings [BuGT2]. Later, Blair, Smith and Sogge proved Strichartz estimates with $s = 4/3p$ both for a compact Riemannian manifold with boundary and for a compact manifold M without boundary, endowed with a Lipschitz metric g [BlSSo1] (see also [BlSSo2], where these results have been recently improved).

Then the spirit of Theorem 1.1 is that, if the initial datum v_0 is spectrally localized in a proper angular sector \mathcal{V} in \mathbf{N}^2 defined as in (1.4), then we are able to prove a Strichartz estimate like (1.9), where the $H^s(S^{2n+1})$ norm of the initial datum at the right-hand side is replaced by the standard non-isotropic norm $W^{s/p}$, for any index s such that $s > 2[1 - 1/(n+1)]$ if $n > 1$ or $s \geq 4/3$ if $n = 1$. Thus there is a gain of $2/(n+1)p$ derivatives ($2/3p$ in the one dimensional case) in comparison to the bounds indicated by non-isotropic Sobolev embeddings. If the initial datum v_0 is spectrally localized in the complementary region, that is, for instance, if $v_0 = \sum_{0 \leq \ell' \leq \ell/M} h_{\ell, \ell'}$, $h_{\ell, \ell'} \in \mathcal{H}^{\ell, \ell'}$, then our techniques only lead to an estimate like (1.9), with $H^s(S^{2n+1})$ norm of the initial datum at the right-hand side replaced by the standard non-isotropic norm $W^{2/p}$, thus providing no improvement with respect to the Sobolev embedding.

The problem of optimality for Strichartz estimates is in general open, also in the Riemannian set-up. As it is well known, the Strichartz estimate proved in [BuGT2] is not sharp, unless in the case $p = 2$, in the class of compact Riemannian manifolds, since J. Bourgain proved that for the flat torus $(\mathbf{R}/2\pi\mathbf{Z})^2$ the Strichartz estimate holds for $p = q = 2$ with loss of ε derivatives, for every $\varepsilon > 0$ [Bou1, Bou2]. Anyway, Burq, Gérard and Tzvetkov were able to improve their intermediate Strichartz estimates in some specific geometries, like spheres and Zoll surfaces [BuGT3, BuGT4]. In our framework, some sharp bounds for the eigenfunctions of the sublaplacian on S^{2n+1} , recently proved by the first author [Ca1, Ca2], do not suffice to prove the optimality in (1.7).

It is worth noticing that different approaches, which have been successfully used in the Riemannian context (we refer in particular to [BuGT3, BuGT2]), are possible and could be used to prove optimal bounds, at least for intermediate (p, q) ; in particular, it would be interesting to prove bilinear and multilinear estimates for spectral projections associated to \mathcal{L} on the complex

sphere, as well as to prove intermediate Strichartz estimates by following the Fourier analytic approach by Bourgain.

We would also like to point out that the compact manifold S^{2n+1} , beyond the pioneering works of G. Folland and D. Geller [Fo1, Ge], has recently attracted a lot of interest in connection with its CR structure; we refer, in particular, to the recent papers [BrFM, BauW, CowKS] and to [Ca1, Ca2, CaP1].

The paper is organized as follows. In Section 2 we start recalling the basic facts about harmonic analysis on the complex sphere. Then we recall the definition of the standard isotropic and non-isotropic Sobolev spaces on the sphere and introduce the mixed Sobolev spaces. In Sections 3 and 4 we prove the basic dispersive estimates for solutions of (1.1) localized at high frequencies. Our proof hinges on a repeated use of the Poisson summation formula, specifically adapted in the key Lemma 4.4 to our case. Following a classical approach, we then deduce in Section 5 the Strichartz estimate (1.7) from the dispersive bounds. Theorem 1.1 is the main tool in Section 6 to prove local well posedness results. Optimality will be discussed in Section 7, where we also make a comment on other possible admissibility conditions.

We shall use the symbol C to denote constants which may vary from one formula to the next, and $[x]$ to denote the greatest integer at most x . The symbol \approx between two positive expressions means that their ratio is bounded above and below.

2. PRELIMINARY FACTS AND NOTATION

In this section we recall some basic facts about spherical harmonics and their relation to the analysis on the complex sphere.

For $n \geq 1$, we denote by \mathbf{C}^{n+1} the $(n+1)$ -dimensional complex space equipped with the scalar product $\langle z, w \rangle := z_1 \bar{w}_1 + \cdots + z_{n+1} \bar{w}_{n+1}$, $z, w \in \mathbf{C}^{n+1}$, and by S^{2n+1} the unit sphere in \mathbf{C}^{n+1} , that is,

$$S^{2n+1} := \{z = (z_1, \dots, z_{n+1}) \in \mathbf{C}^{n+1} : \langle z, z \rangle = 1\}.$$

The sphere S^{2n+1} is a strongly pseudoconvex CR manifold and thus endowed with subriemannian structure. Its Carnot-Carathéodory distance is equivalent to the so-called *Korányi distance* d ,

$$d(z, w) := |1 - \langle z, w \rangle|^{1/2}, \quad (2.1)$$

$z, w \in S^{2n+1}$, see [Na].

The *homogeneous dimension* Q of S^{2n+1} , that will play a relevant rôle in our analysis, is given by $Q := 2n + 2$, since it is well known that $\text{Vol}(B(z, r)) \sim r^Q$, where $B(z, r)$ denotes the ball centered at $z \in S^{2n+1}$ with radius $r > 0$.

2.1. Spherical harmonics and spectral projections. Consider the space $L^2(S^{2n+1})$, equipped with the inner product

$$(f, g) := \int_{S^{2n+1}} f(z) \overline{g(z)} d\sigma(z),$$

where $d\sigma$ is the Lebesgue surface measure, which is invariant under the action of the unitary group $U(n+1)$.

For non-negative integers ℓ, ℓ' , $\mathcal{H}^{\ell, \ell'}$ is the vector space of the restrictions to S^{2n+1} of harmonic polynomials $p(z, \bar{z})$, homogeneous of degree ℓ in z and of degree ℓ' in \bar{z} . A function in $\mathcal{H}^{\ell, \ell'}$ is called a *complex spherical harmonic of bidegree* (ℓ, ℓ') .

When $\ell' = 0$, the space $\mathcal{H}^{\ell, 0}$ consists of holomorphic polynomials, and $\mathcal{H}^{0, \ell}$ consists of polynomials whose complex conjugates are holomorphic.

The subspaces $\mathcal{H}^{\ell, \ell'}$ have finite dimension $d_{\ell, \ell'}$ given by

$$d_{\ell, \ell'} := n \frac{\ell + \ell' + n}{\ell \ell'} \binom{\ell + n - 1}{\ell - 1} \binom{\ell' + n - 1}{\ell' - 1} \quad (2.2)$$

if $\ell, \ell' \geq 1$, and by

$$d_{\ell, 0} = d_{0, \ell} := \binom{\ell + n}{\ell},$$

if ℓ or ℓ' equals 0.

Moreover, the subspaces $\mathcal{H}^{\ell, \ell'}$ are $U(n+1)$ -invariant, pairwise orthogonal and their sum is dense in $L^2(S^{2n+1})$; more explicitly, if we denote by the symbol $\pi_{\ell, \ell'}$ the orthogonal projector mapping $L^2(S^{2n+1})$ onto $\mathcal{H}^{\ell, \ell'}$, then each function $f \in L^2(S^{2n+1})$ may be decomposed in a unique way as

$$f = \sum_{\ell, \ell'=0}^{+\infty} \pi_{\ell, \ell'} f, \quad (2.3)$$

where the series converges unconditionally to f in the L^2 -topology.

A special rôle in $\mathcal{H}^{\ell, \ell'}$ is played by the so-called *zonal* functions. Let $\{Y_k^{\ell, \ell'}\}$, $k = 1, \dots, d_{\ell, \ell'}$, be an orthonormal basis for $\mathcal{H}^{\ell, \ell'}$. For $(z, w) \in S^{2n+1} \times S^{2n+1}$ set

$$Z_{\ell, \ell'}(z, w) := \sum_{k=1}^{d_{\ell, \ell'}} Y_k^{\ell, \ell'}(z) \overline{Y_k^{\ell, \ell'}(w)}.$$

Then, for all $f \in \mathcal{H}^{\ell, \ell'}$ we have

$$f(z) = \int_{S^{2n+1}} f(w) Z_{\ell, \ell'}(z, w) d\sigma(w). \quad (2.4)$$

Since $\mathcal{H}^{\ell, \ell'}$ is finite dimensional, the above pairing makes sense for all $f \in L^2(S^{2n+1})$.

For each fixed point $w \in S^{2n+1}$, the function $f(w) = Z_{\ell, \ell'}(\cdot, w)$ is in $\mathcal{H}^{\ell, \ell'}$ and it is constant on the orbits of the stabilizer of w in $U(n+1)$, which is isomorphic to $U(n)$. In other words $Z_{\ell, \ell'}(z, w)$ depends only on $\langle z, w \rangle$, and we write

$$\langle z, w \rangle = e^{i\omega} \cos \theta, \quad \theta \in [0, \pi/2], \quad \omega \in [0, 2\pi). \quad (2.5)$$

With an abuse of notation, we will also denote by $Z_{\ell, \ell'}$ the function depending on the 1-dimensional complex variable $\langle z, w \rangle$, that is,

$$Z_{\ell, \ell'}(\langle z, w \rangle) = Z_{\ell, \ell'}(z, w).$$

An explicit formula for the zonal function $Z_{\ell, \ell'} \in \mathcal{H}^{\ell, \ell'}$, for $\ell' \geq \ell \geq 1$, is given by

$$Z_{\ell, \ell'}(e^{i\omega} \cos \theta) = \frac{d_{\ell, \ell'}}{\omega_{2n+1}} \frac{\ell!(n-1)!}{(\ell+n-1)!} \times e^{i\omega(\ell'-\ell)} (\cos \theta)^{\ell'-\ell} P_{\ell}^{(n-1, \ell'-\ell)}(\cos 2\theta), \quad (2.6)$$

where ω_{2n+1} denotes the surface area of S^{2n+1} and $P_{\ell}^{(n-1, \ell'-\ell)}$ is the Jacobi polynomial, see [Sz].

For the case $\ell' < \ell$, it suffices to recall that $Z_{\ell, \ell'}(z, w) = \overline{Z_{\ell', \ell}(w, z)}$.

Since $P_0^{(n-1, \ell)} \equiv 1$, if $\ell' = 0$ the zonal function is given by

$$Z_{\ell, 0}(z, w) = \frac{1}{\omega_{2n+1}} \binom{\ell+n}{\ell} \overline{\langle z, w \rangle}^{\ell}.$$

The following bound for the zonal functions is well known, and appears in [Fo3]. For any $z, w \in S^{2n+1}$ we have

$$|Z_{\ell, \ell'}(z, w)| \leq \frac{d_{\ell, \ell'}}{\omega_{2n+1}}. \quad (2.7)$$

Finally, it is easy to check that the orthogonal projector $\pi_{\ell, \ell'}$ may be written as

$$\pi_{\ell, \ell'} f(z) = \int_{S^{2n+1}} f(w) Z_{\ell, \ell'}(z, w) d\sigma(w).$$

2.2. Classical Sobolev spaces. The algebra of $U(n)$ -invariant differential operators on S^{2n+1} is commutative and generated by two elements: a basis is given by the Laplace–Beltrami operator Δ and by the sublaplacian \mathcal{L} , defined by (1.2). Both operators are self-adjoint and positive definite. \mathcal{L} turns out to be subelliptic. Moreover, each subspace $\mathcal{H}^{\ell, \ell'}$ is an eigenspace both for Δ with eigenvalue $\mu_{\ell, \ell'} := (\ell + \ell')(\ell + \ell' + 2n)$, and for \mathcal{L} with eigenvalue $\lambda_{\ell, \ell'} := 2\ell\ell' + n(\ell + \ell')$. For these and other properties of \mathcal{L} we refer the reader to [Ge] and [RU].

The non-isotropic Sobolev spaces on the complex sphere can be defined in terms of suitable powers of $I + \mathcal{L}$, or, equivalently, in terms of suitable powers of the conformal sublaplacian $\mathcal{D} := \mathcal{L} + \frac{n^2}{2}$; see, for instance, [Fo2]. More precisely, for $1 \leq p \leq \infty$ we set

$$W^{r, p}(S^{2n+1}) := \{f \in L^p(S^{2n+1}) : (I + \mathcal{L})^{r/2} f \in L^p\}. \quad (2.8)$$

The operator $(I + \mathcal{L})^{r/2}$ can be defined locally transferring the analogous operator from the Heisenberg group via the Cayley transform, see [Fo2, § 3], and also [CaP2].

We will mostly deal with the case of L^2 -integrability, and we simply write W^r for $W^{r, 2}$. For functions in W^r we have the identity

$$(I + \mathcal{L})^{r/2} f = \sum_{\ell, \ell'=0}^{+\infty} (1 + \lambda_{\ell, \ell'})^{r/2} \pi_{\ell, \ell'} f$$

Then, W^r is a Hilbert space under the inner product

$$(f, g)_{W^r} := \int_{S^{2n+1}} (I + \mathcal{L})^{r/2} f \overline{(I + \mathcal{L})^{r/2} g}.$$

For $s \geq 0$, we shall denote by $H^s(S^{2n+1})$ the classical Sobolev space on S^{2n+1} , defined as in (2.8), with the operator $I + \mathcal{L}$ replaced by the operator $I + \Delta$. In particular, H^s is endowed with the norm

$$\|f\|_{H^s} = \left(\sum_{\ell, \ell'=0}^{\infty} (1 + \mu_{\ell, \ell'})^s \|\pi_{\ell, \ell'} f\|_{L^2}^2 \right)^{1/2}.$$

The following inclusions follow

$$H^s \subseteq W^s \subseteq H^{s/2}.$$

For both isotropic and non-isotropic Sobolev immersion theorems in a CR setting we refer to the seminal papers [Fo2] and [FoSt], where results are proved in the framework of Heisenberg groups. Anyway, it is not difficult to check that the same inclusions hold on complex spheres.

2.3. Mixed Sobolev spaces. We now introduce a family of Sobolev-type spaces that measure the regularity of functions differently according to their spectral localization.

Fix a constant $M > 1$ and define the proper cone $\mathcal{V} = \mathcal{V}_M$ in \mathbf{N}^2

$$\mathcal{V} := \{(\ell, \ell') : \ell/M < \ell' < M\ell\} \quad (2.9)$$

and the pair of edges $\mathcal{E} = \mathcal{E}_M$

$$\mathcal{E} := \{(\ell, \ell') : \ell' \leq \ell/M \text{ or } \ell' \geq M\ell\}. \quad (2.10)$$

We define the corresponding spectral projections

$$\pi_{\mathcal{V}} = \sum_{\ell/M < \ell' < M\ell} \pi_{\ell, \ell'} \quad \text{and} \quad \pi_{\mathcal{E}} = \sum_{\ell' \leq \ell/M \text{ or } \ell' \geq M\ell} \pi_{\ell, \ell'}.$$

We then introduce the corresponding spaces of spectrally localized functions

$$L_{\mathcal{V}}^2(S^{2n+1}) = \{u \in L^2(S^{2n+1}) : u = \pi_{\mathcal{V}}u\}$$

and

$$L_{\mathcal{E}}^2(S^{2n+1}) = \{u \in L^2(S^{2n+1}) : u = \pi_{\mathcal{E}}u\}.$$

We define the mixed Sobolev spaces $\mathcal{X}^{(r,s)} = \mathcal{X}^{(r,s)}(S^{2n+1})$ as

$$\mathcal{X}^{(r,s)} = \{u \in L^2(S^{2n+1}) : \pi_{\mathcal{V}}u \in W^r(S^{2n+1}) \text{ and } \pi_{\mathcal{E}}u \in W^s(S^{2n+1})\}, \quad (2.11)$$

with norm given by

$$\|u\|_{\mathcal{X}^{(r,s)}} = \left(\sum_{(\ell, \ell') \in \mathcal{V}} (1 + \lambda_{\ell, \ell'})^r \|\pi_{\ell, \ell'}u\|_{L^2}^2 + \sum_{(\ell, \ell') \in \mathcal{E}} (1 + \lambda_{\ell, \ell'})^s \|\pi_{\ell, \ell'}u\|_{L^2}^2 \right)^{1/2}$$

Notice that the norm depends on M , and that different values of M give rise to equivalent norms.

In general, given a function space $\mathcal{Y} \subseteq L^2(S^{2n+1})$, we denote by $\mathcal{Y}_{\mathcal{V}}$ and $\mathcal{Y}_{\mathcal{E}}$ respectively, the subspaces of \mathcal{Y} of the functions that are spectrally localized in \mathcal{V} and \mathcal{E} , respectively. Then, we have

$$\mathcal{X}^{(r,s)} = W_{\mathcal{V}}^r \cap W_{\mathcal{E}}^s.$$

For the mixed Sobolev spaces $\mathcal{X}^{(r,s)}$ we have the following elementary result that gives embedding in the Lebesgue spaces and comparison with the non-isotropic and classical Sobolev spaces.

Proposition 2.1. *Let $M > 1$ be fixed. Given $r, s \geq 0$ the following properties hold true.*

(1) *If $\min(r, s) > Q(\frac{1}{2} - \frac{1}{q})$ then for all $u \in \mathcal{C}^\infty(S^{2n+1})$ we have*

$$\|u\|_{L^q} \leq C\|u\|_{\mathcal{X}^{(r,s)}}.$$

(2) *If $u \in \mathcal{C}_{\mathcal{V}}^\infty$, then*

$$\|u\|_{\mathcal{X}^{(r,s)}} \approx \|u\|_{W^r} \approx \|u\|_{H^r},$$

and, for $\min(r, s) > (2n+1)(\frac{1}{2} - \frac{1}{q})$

$$\|u\|_{L^q} \leq C\|u\|_{\mathcal{X}^{(r,s)}}.$$

(3) *For all $u \in \mathcal{C}^\infty$ such that $\pi_{\ell, \ell'}u = 0$ for $\min(\ell, \ell') > M$ we have*

$$\|u\|_{\mathcal{X}^{(r,s)}} = \|u\|_{W^r} \approx \|u\|_{H^{r/2}}.$$

The constants involved in the above estimates depend on M .

Proof. (1) It suffices to recall the embedding theorems for the non-isotropic Sobolev spaces W^r . Thm. 5.15 in [Fo2] entails that, if $u \in \mathcal{C}_V^\infty$, then $\|u\|_{L^q} \leq C\|u\|_{W^{r,2}}$ if $r > Q(\frac{1}{2} - \frac{1}{q})$. The result now follows easily.

Next, for $u \in \mathcal{C}_V^\infty$ we have

$$\begin{aligned} \|u\|_{(r,s)}^2 &\approx \sum_{\ell/M < \ell' < M\ell} (1 + \lambda_{\ell,\ell'})^r \|\pi_{\ell,\ell'} u\|_2^2 \approx \sum_{\ell/M < \ell' < M\ell} (1 + \ell)^{2r} \|\pi_{\ell,\ell'} u\|_2^2 \\ &\approx \sum_{\ell/M < \ell' < M\ell} (1 + \mu_{\ell,\ell'})^r \|\pi_{\ell,\ell'} u\|_2^2. \end{aligned}$$

The second statement in (2) now follows from the classical embedding theorem for the Sobolev space $H^\sigma(S^{2n+1})$.

Finally, (3) follows at once since, for $\min(\ell, \ell') \leq M$, $(1 + \lambda_{\ell,\ell'}) \approx (1 + \mu_{\ell,\ell'})^{1/2}$. \square

2.4. Littlewood–Paley decomposition. From a technical point of view, an essential tool to prove the Strichartz estimate (1.7) will be an estimate that is an easy consequence of the Littlewood–Paley decomposition for the sublaplacian \mathcal{L} on the complex sphere. More precisely, we shall use the following result.

Theorem 2.2. *Let $\tilde{\psi} \in \mathcal{C}_0^\infty(\mathbf{R}_+)$ and $\psi \in \mathcal{C}_0^\infty(\mathbf{R})$, such that*

$$\tilde{\psi}(\lambda) + \sum_{j=1}^{\infty} \psi(2^{-2j}\lambda) = 1, \quad \lambda \in \mathbf{R}. \quad (2.12)$$

Then for $2 \leq q < \infty$ there exists a constant C_q such that

$$\|f\|_{L^q(S^{2n+1})} \leq C_q \left(\|\tilde{\psi}(\mathcal{L})f\|_{L^q(S^{2n+1})} + \left(\sum_{j=1}^{+\infty} \|\psi(2^{-2j}\mathcal{L})f\|_{L^q(S^{2n+1})}^2 \right)^{1/2} \right), \quad (2.13)$$

for $f \in L^q(S^{2n+1})$.

This result and the Littlewood–Paley decomposition are of independent interest (see the recent paper [Bouc] for a discussion of analogous inequalities in the Riemannian case), and are proved in the forthcoming paper [CaP2].

3. THE DISPERSIVE ESTIMATE

In this section we study the dispersive properties for solutions of the Schrödinger equation that are spectrally localized.

The solutions of (1.1) do not satisfy in general a dispersive estimate, either globally (as constant solutions show for large t) or locally in time. A similar lack of the dispersive effect was noticed by Burk, Gérard and Tzvetkov in the Riemannian case on compact manifolds as well (see [BuGT2]).

Indeed, if we could prove a dispersive estimate of the form

$$\|e^{it\mathcal{L}}\|_{(L^1(S^{2n+1}), L^\infty(S^{2n+1}))} \leq \frac{C}{|t|^{q_0}}$$

for some $q_0 > 0$ and for some $t > 0$, then the L^∞ norm of eigenfunctions of the sublaplacian should be controlled by the L^1 norm and this is not true in general (see Theorem 3.1 in [Ca2], where bounds are proved for the L^1 norm of zonal functions).

However, it is possible to prove a family of dispersive estimates on small time intervals related to the frequencies of the data, that will suffice for the proof of the Strichartz estimates (see [BaCh] and [Tat] for a first application of this idea).

In this paper, we are able to prove different dispersive estimates for data v that are spectrally localized according to a *double* decomposition of the spectrum.

We introduce a spectral cut-off. Let φ be a non-negative smooth function with support contained in the interval $[a, b]$, with $0 < a < b < \infty$. and $h \in (0, 1]$, we consider the operator

$$\varphi(h^2 \mathcal{L}) : L^2(S^{2n+1}) \rightarrow L^2(S^{2n+1}),$$

defined by the functional calculus for the sublaplacian.

Next, we fix a smooth cut-off function ψ with compact support in $[1/M, M]$, where $M > 1$ is a (large) constant.

Theorem 3.1. *Let φ, ψ be smooth cut-off functions be defined as above. Let p, p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$, $p \in [1, 2]$. Then the following estimates hold.*

- (i) *Let s_n is as in (1.6) and let $s > s_n$ if $n > 1$, or $s \geq s_n$ if $n = 1$. Then there exist $c, C_s > 0$ such that for all $v_0 \in C^\infty(S^{2n+1})$, for all $h \in (0, 1]$*

$$\left\| \sum_{\ell, \ell' \geq 0} e^{it\lambda_{\ell, \ell'}} \varphi(h^2 \lambda_{\ell, \ell'}) \psi(\ell'/\ell) \pi_{\ell, \ell'}(v_0) \right\|_{L^{p'}(S^{2n+1})} \leq \frac{C_s}{|t|^{Q/2(\frac{1}{p} - \frac{1}{p'})}} \|v_0\|_{L^p(S^{2n+1})} \quad (3.1)$$

for all $t \in I_s := [-ch^s, ch^s]$.

- (ii) *Then there exists $C > 0$ such that for all $v_0 \in C^\infty(S^{2n+1})$, for all $h \in (0, 1]$*

$$\left\| \sum_{\ell, \ell' \geq 0} e^{it\lambda_{\ell, \ell'}} \varphi(h^2 \lambda_{\ell, \ell'}) (1 - \psi(\ell'/\ell)) \pi_{\ell, \ell'}(v_0) \right\|_{L^{p'}(S^{2n+1})} \leq \frac{C_s}{|t|^{Q/2(\frac{1}{p} - \frac{1}{p'})}} \|v_0\|_{L^p(S^{2n+1})} \quad (3.2)$$

for all $t \in I_2 := [-h^2, h^2]$.

Remark 3.2. We point out that the index s which determines the length of the time interval I_s in (3.1) and (3.2), is subject to the following upper bound. For, the kernel of the operator $e^{it\mathcal{L}}\varphi(h^2\mathcal{L})$ is given by

$$K_h(t, z, w) = \sum_{\ell, \ell' = 0}^{\infty} e^{it\lambda_{\ell, \ell'}} \varphi(h^2 \lambda_{\ell, \ell'}) Z_{\ell, \ell'}(z, w).$$

Reasoning as in [BuGT2], we have

$$\begin{aligned} \|e^{it\mathcal{L}}\varphi(h^2\mathcal{L})\|_{(L^1, L^\infty)} &= \|K_h(t, \cdot, \cdot)\|_{L^\infty(S^{2n+1} \times S^{2n+1})} \geq C \|K_h(t, \cdot, \cdot)\|_{L^2(S^{2n+1} \times S^{2n+1})} \\ &\geq C \left(\sum_{\ell, \ell' = 0}^{\infty} |\varphi(h^2 \lambda_{\ell, \ell'})|^2 d_{\ell, \ell'} \right)^{1/2} \geq \frac{C}{h^{n-1}} \left(\sum_{\lambda_{\ell, \ell'} \sim h^{-2}} (\ell + \ell') \right)^{1/2} \\ &\geq \frac{C}{h^{n-1}} \left(\sum_{\ell \sim h^{-2}} \ell \right)^{1/2} = \frac{C}{h^{n+1}} \end{aligned}$$

where in particular we used (2.2). Then estimates like (3.1) or (3.2) for $p = 1$ imply $|t| \leq ch$ for some $c > 0$, that is, $s \geq 1$.

Proof of Theorem 3.1. For all $t \in \mathbf{R}$ we have

$$\|e^{it\mathcal{L}}\varphi(h^2\mathcal{L})v_0\|_{L^2} \leq C\|v_0\|_{L^2}.$$

Thus, as a consequence of the Riesz-Thorin Theorem and Young's inequality, it suffices to prove the following estimates

$$\left\| \sum_{\ell, \ell' \geq 0} e^{it\lambda_{\ell, \ell'}} \varphi(h^2\lambda_{\ell, \ell'}) \psi(\ell'/\ell) Z_{\ell, \ell'} \right\|_{L^\infty(S^{2n+1} \times S^{2n+1})} \leq \frac{C}{|t|^{Q/2}} \quad (3.3)$$

for all $|t| \leq h^s$, where $s > s_n := 2[1 - 1/(n+1)]$ and

$$\left\| \sum_{\ell, \ell' \geq 0} e^{it\lambda_{\ell, \ell'}} \varphi(h^2\lambda_{\ell, \ell'}) (1 - \psi(\ell'/\ell)) Z_{\ell, \ell'} \right\|_{L^\infty(S^{2n+1} \times S^{2n+1})} \leq \frac{C}{|t|^{Q/2}} \quad (3.4)$$

for all $|t| \leq h^2$.

In order to prove Theorem 3.1 we break the proof of (3.3) and (3.4) into a few steps, that now we summarize.

Step 1. We prove both estimates when (i) $h \geq \varepsilon_0$, where $\varepsilon_0 > 0$ is a fixed constant and (ii) when $|t| \leq h^2$. This second case proves in fact that both (3.3) and (3.4) hold for these values of t and in particular establishes (3.2) in Theorem 3.1.

Step 2. Recalling (2.6) and (2.5), we prove (3.3) when $\langle z, w \rangle = e^{i\omega} \cos \theta$ varies in a fixed compact set of the unit disk, that is, when $\theta \in [\varepsilon_1, \pi/2]$, for some $\varepsilon_1 > 0$.

Step 3. Next, we assume that $h^2 \leq |t| \leq h^s$, and $0 < h < \varepsilon_0$ and prepare to estimate

$$\left| \sum_{\ell, \ell'=1}^{+\infty} e^{it\lambda_{\ell, \ell'}} \varphi(h^2\lambda_{\ell, \ell'}) \psi(\ell'/\ell) Z_{\ell, \ell'}(z, w) \right|$$

when $\langle z, w \rangle$ varies outside the compact set fixed in Step 2, that is, when $\theta < \varepsilon_1$.

First we need to distinguish between the diagonal case $\ell = \ell'$ and the sums over $\ell > \ell'$ and $\ell < \ell'$. To do this, we introduce an even cut-off function η_0 , identically 1 for $|\xi| \leq 1/4$ and identically 0 for $|\xi| \geq 1/2$, and the two cut-off functions $\eta_\pm(\xi) = \chi_{(0, +\infty)}[1 - \eta_0(\pm\xi)]$, supported respectively on $\ell > \ell'$ and $\ell < \ell'$. Accordingly, we decompose the sum above as

$$K^0 + K^+ + K^-$$

by writing $1 = \eta_0(\ell' - \ell) + \eta_+(\ell' - \ell) + \eta_-(\ell' - \ell)$.

The estimate for K^0 turns out to be trivial, while, in order to estimate K^\pm , we need another decomposition. Clearly, it suffices to consider the case of K^+ , which is supported when $\ell' > \ell$.

We need to distinguish between the cases $\theta \leq 1/\ell'$ (recall that $\ell' = \max\{\ell', \ell\}$) and $\theta > 1/\ell'$. Thus we introduce a cut-off function χ_1 supported in $[0, 2]$, set $\chi_2 = 1 - \chi_1$ and split K^+ as $K_1^+(\omega, \theta) + K_2^+(\omega, \theta)$, where, for $j = 1, 2$,

$$K_j^+(\omega, \theta) := \sum_{\ell, \ell'=1}^{+\infty} e^{it\lambda_{\ell, \ell'}} \varphi(h^2\lambda_{\ell, \ell'}) \psi(\ell'/\ell) \eta_+(\ell' - \ell) \chi_j((\ell + \ell' + n)\theta) Z_{\ell, \ell'}(z, w).$$

Step 4. Here we prove the estimate for $K_1^\pm(\omega, \theta)$.

Step 5. Finally, we prove the estimate for $K_2^\pm(\omega, \theta)$ and complete the proof of Theorem 3.1.

Step 1. We begin proving that the estimate (3.3) is trivial in two cases: when $h \geq \varepsilon_0$ and in the low frequency case, that is, when $|t| \leq h^2$. In this case, the oscillations of the exponential function are ineffective.

In [CaP1] the authors proved a restriction-type lemma for blocks of spectral projections associated to the sublaplacian on S^{2n+1} . A key ingredient in the proof was the following estimate.

Lemma 3.3. *Let $1 \leq a < b$ be fixed. Then there exists a constant $C > 0$ depending only on n such that*

$$\sum_{\lambda_{\ell, \ell'} \in (a, b]} (\ell + \ell') \leq Cb(b - a + \log(b + 1)). \quad (3.5)$$

Lemma 3.4. *There exists a constant $C > 0$ such that the estimates (3.3) and (3.4) hold in the following cases:*

- (i) when $t \in I_2$;
- (ii) when $h \geq \varepsilon_0$, for all $t \in I_s$;

where I_2 and I_s are defined in Theorem 3.1.

Proof. For $z, w \in S^{2n+1}$, $|t| \leq h^2$, using (2.7) and Lemma 3.3 we have

$$\begin{aligned} \left| \sum_{a/h^2 < \lambda_{\ell, \ell'} < b/h^2} e^{it\lambda_{\ell, \ell'}} \varphi(h^2 \lambda_{\ell, \ell'}) Z_{\ell, \ell'}(z, w) \right| &\leq \sum_{a/h^2 < \lambda_{\ell, \ell'} < b/h^2} \frac{d_{\ell, \ell'}}{\omega_{2n+1}} \leq \sum_{a/h^2 < \lambda_{\ell, \ell'} < b/h^2} \lambda_{\ell, \ell'}^{n-1} (\ell + \ell') \\ &\leq \frac{C}{h^{2n-2}} \sum_{a/h^2 < \lambda_{\ell, \ell'} < b/h^2} (\ell + \ell') \\ &\leq \frac{C}{|h|^{2n+2}}, \end{aligned}$$

for a suitable positive constant C .

If $|t| \leq h^2$, conclusion (i) follows at once. If $h \geq \varepsilon_0$, (ii) also follows at once, since

$$h^{-(2n+2)} \leq \varepsilon_0^{-2} h^{-2n} \leq \varepsilon_0^{-2} |t|^{-Q/2}.$$

This proves the lemma. \square

Observe that this lemma in particular proves the estimate contained in (3.4)– in fact this is the trivial part of the estimate, and it does provide no improvement with respect to the Sobolev embedding theorem.

Step 2. The dispersive estimate (3.1) follows easily also when $\langle z, w \rangle$ varies in a compact subset of the unit disk, as a consequence of the following result. First we recall the following estimates for Jacobi polynomials, see e.g. [BoCl, page 231],

$$\left| P_{\ell}^{(\alpha, \beta)}(\cos \theta) \right| \leq \begin{cases} C\ell^{\alpha} & \text{if } 0 \leq \theta \leq \frac{\pi}{2}, \\ C\ell^{-1/2}\theta^{-\alpha-\frac{1}{2}} & \text{if } 0 < \theta \leq \frac{\pi}{2}, \\ C\ell^{-1/2}|\pi - \theta|^{-\beta-\frac{1}{2}} & \text{if } \frac{\pi}{2} \leq \theta < \pi, \\ C\ell^{\beta} & \text{if } \frac{\pi}{2} \leq \theta \leq \pi. \end{cases} \quad (3.6)$$

Lemma 3.5. *Let $0 < \varepsilon_1 < \frac{\pi}{2}$ be fixed and set*

$$\mathcal{K}_{\varepsilon_1} := \{(z, w) \in S^{2n+1} \times S^{2n+1} : \langle z, w \rangle = e^{i\omega} \cos \theta, \omega \in [0, 2\pi], \theta \in [\varepsilon_1, \pi/2]\}.$$

Then, there exists $C > 0$ such that

$$\sup_{(z, w) \in \mathcal{K}_{\varepsilon_1}} \left| \sum_{\ell, \ell' \geq 0} e^{it\lambda_{\ell, \ell'}} \varphi(h^2 \lambda_{\ell, \ell'}) Z_{\ell, \ell'}(z, w) \right| \leq \frac{C}{|t|^{Q/2}}. \quad (3.7)$$

Proof. The proof is simple since again in this case we do not need to consider the oscillations of the kernel. By symmetry in the parameters ℓ and ℓ' , in (3.7) it suffices to consider the case $\ell' \geq \ell$.

Assume first that $\theta \in [\varepsilon_1, \frac{\pi}{4}]$. In this case, for $(z, w) \in \mathcal{K}_{\varepsilon_1}$, we have, using the first inequality in (3.6) and (2.2)

$$\begin{aligned}
& \left| \sum_{\ell, \ell' \geq 1} e^{it\lambda_{\ell, \ell'}} \varphi(h^2 \lambda_{\ell, \ell'}) Z_{\ell, \ell'}(z, w) \right| \\
& \leq C \sum_{\ell, \ell' \geq 1} \frac{d_{\ell, \ell'}}{\ell^{n-1}} (\cos \theta)^{\ell' - \ell} \left| P_{\ell}^{(n-1, \ell' - \ell)}(\cos 2\theta) \right| \\
& \leq C \sum_{a/h^2 \leq \lambda_{\ell, \ell'} \leq b/h^2} \frac{d_{\ell, \ell'}}{\ell^{n-1}} (\cos \theta)^{\ell' - \ell} \ell^{n-1} \\
& \leq \frac{C}{h^{2n-2}} \sum_{a/h^2 \leq \lambda_{\ell, \ell'} \leq b/h^2} (\ell + \ell') (\cos \theta)^{\ell' - \ell} \\
& \leq \frac{C}{h^{2n-2}} \left(\sum_{\ell=0}^{\lfloor c/h \rfloor} \sum_{\ell' - \ell = 0}^{\lfloor c/h \rfloor} (\ell' - \ell) (\cos \theta)^{\ell' - \ell} + \sum_{\ell=0}^{\lfloor c/h \rfloor} 2\ell \sum_{\ell' - \ell = 0}^{\lfloor c/h \rfloor} (\cos \theta)^{\ell' - \ell} \right) \\
& \leq \frac{C}{h^{2n}},
\end{aligned}$$

since $\theta \in [\varepsilon_1, \pi/4]$.

Next, when $\theta \in [\pi/4, \pi/2]$, we observe that the sum vanishes when $\theta = \pi/2$ and then we split it into two parts. Recalling that we are assuming $\ell \leq \ell'$ we set

$$\begin{aligned}
E_1 &= \{(\ell, \ell') \in \mathbf{N}^2 : a/h^2 < \lambda_{\ell, \ell'} < b/h^2, \ell > 1/|\pi - 2\theta|\}, \\
E_2 &= \{(\ell, \ell') \in \mathbf{N}^2 : a/h^2 < \lambda_{\ell, \ell'} < b/h^2, \ell \leq 1/|\pi - 2\theta|\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
& \left| \sum_{\ell' \geq \ell \geq 1} e^{it\lambda_{\ell, \ell'}} \varphi(h^2 \lambda_{\ell, \ell'}) Z_{\ell, \ell'}((z, w)) \right| \\
& \leq \sum_{(\ell, \ell') \in E_1} \varphi(h^2 \lambda_{\ell, \ell'}) |Z_{\ell, \ell'}((z, w))| + \sum_{(\ell, \ell') \in E_2} \varphi(h^2 \lambda_{\ell, \ell'}) |Z_{\ell, \ell'}((z, w))| \\
& =: S_1 + S_2.
\end{aligned}$$

Using the last inequality in (3.6) we have

$$\begin{aligned}
S_1 &\leq C \sum_{(\ell, \ell') \in E_1} \frac{d_{\ell, \ell'}}{\ell^{n-1}} (\cos \theta)^{\ell' - \ell} \left| P_{\ell}^{(n-1, \ell' - \ell)}(\cos 2\theta) \right| \\
&\leq C \sum_{(\ell, \ell') \in E_1} \frac{d_{\ell, \ell'}}{\ell^{n-1}} (\pi/2 - \theta)^{\ell' - \ell} \ell^{\ell' - \ell} \\
&\leq C \sum_{(\ell, \ell') \in E_1} 2^{-(\ell' - \ell)} \frac{d_{\ell, \ell'}}{\ell^{n-1}}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{C}{h^{2n-2}} \sum_{(\ell, \ell') \in E_1} 2^{-(\ell' - \ell)} \frac{(\ell + \ell')}{\ell^{n-1}} \\
&\leq \frac{C}{h^{2n-2}} \sum_{\ell=0}^{\lfloor c/h \rfloor} \sum_{\ell' - \ell = 0}^{\lfloor c/h \rfloor} \left(\frac{1}{\ell^{n-1}} (\ell' - \ell) 2^{-(\ell' - \ell)} + \frac{1}{\ell^{n-2}} 2^{-(\ell' - \ell)} \right) \\
&\leq \frac{C}{h^{2n}}.
\end{aligned}$$

An analogous bound may be proved for S_2 using the third inequality in (3.6), we finally obtain (3.7). \square

4. THE MAIN ESTIMATE

Step 3. Now we turn to the estimate (3.1). As a consequence of what has been proved in Steps 1-2, from now on we may assume that h is sufficiently small, and precisely that $0 < h < \varepsilon_0$, and that $t \in A$ where

$$A := \{t : h^2 \leq |t| \leq ch^s\}, \quad (4.1)$$

Recall also that, because of the presence of the cut-off function ψ in (3.1), we may consider the parameters ℓ, ℓ' to be such that $1/M < \ell'/\ell \leq M$, where $M > 1$ is a fixed (large) constant.

Starting from (3.3) we now wish to show that for every $\kappa > 0$ there exists $C > 0$ such that

$$\sup_{(z, w) \in \Omega} \left| \sum_{\ell, \ell'=1}^{+\infty} e^{it\lambda_{\ell, \ell'}} \varphi(h^2 \lambda_{\ell, \ell'}) \psi(\ell'/\ell) Z_{\ell, \ell'}(z, w) \right| \leq C \frac{1}{h^{2n+\kappa}}, \quad (4.2)$$

for all $t \in A$, where

$$\Omega = \{(z, w) \in S^{2n+1} \times S^{2n+1} : \langle z, w \rangle e^{i\omega} \cos \theta, \text{ where } 0 \leq \theta \leq \varepsilon_1, \omega \in [0, 2\pi)\}.$$

We shall need to differentiate the proof between the cases $n = 1$ and $n > 1$ only at the end of Step 5, so that we will not distinguish between different values of n until Proposition 4.11.

We notice that we may assume $t > 0$, since passing to the complex conjugate in (4.2) would change $Z_{\ell, \ell'}$ into $Z_{\ell', \ell}$.

It turns out to be convenient to further simplify the problem, by separating the cases $\ell = \ell'$, $\ell < \ell'$ and $\ell > \ell'$.

We can do this by introducing yet another cut-off function. We let η_0 be an even cut-off function, identically 1 for $|\xi| \leq 1/4$ and identically 0 for $|\xi| \geq 1/2$. Then we write $1 = \eta_0(\xi) + \eta_-(\xi) + \eta_+(\xi)$, where $\eta_{\pm}(\xi) = (1 - \eta_0(\xi)) \chi_{[0, +\infty)}(\pm \xi)$. Accordingly, we decompose the sum in (4.2) as

$$K^0 + K^+ + K^-. \quad (4.3)$$

It is easy to estimate K^0 , since it coincides with the sum in (4.2) restricted to the diagonal terms $\ell = \ell'$ and in this case the sum reduces to a summation in one variable.

Lemma 4.1. *For all $t \in A$ we have*

$$\sup_{(z, w) \in \Omega} \left| \sum_{\ell, \ell'=1}^{+\infty} e^{it\lambda_{\ell, \ell'}} \varphi(h^2 \lambda_{\ell, \ell'}) \eta_0(\ell - \ell') Z_{\ell, \ell'}(z, w) \right| \leq C \frac{1}{h^{2n}}. \quad (4.4)$$

Proof. We easily check that

$$\begin{aligned} \sup_{(z,w) \in \Omega} \left| \sum_{\ell, \ell'=1}^{+\infty} e^{it\lambda_{\ell, \ell'}} \varphi(h^2 \lambda_{\ell, \ell'}) \eta_0(\ell - \ell') Z_{\ell, \ell'}(z, w) \right| &= \sup_{(z,w) \in \Omega} \left| \sum_{\ell=1}^{+\infty} e^{it\lambda_{\ell, \ell}} \varphi(h^2 \lambda_{\ell, \ell}) Z_{\ell, \ell}(z, w) \right| \\ &\leq C \left| \sum_{\ell=1}^{+\infty} \varphi(h^2 \lambda_{\ell, \ell}) (\lambda_{\ell, \ell})^{n-1} \ell \right| \leq C \frac{1}{h^{2n}} \end{aligned}$$

for all $t \in A$. □

We are left with the estimate of $K^\pm(\omega, \theta)$, where

$$K^\pm(\omega, \theta) := \sum_{\ell, \ell'=0}^{\infty} e^{it\lambda_{\ell, \ell'}} \varphi(h^2 \lambda_{\ell, \ell'}) \eta_\pm(\ell' - \ell) \psi(\ell'/\ell) Z_{\ell, \ell'}(z, w). \quad (4.5)$$

Our proof of (4.2), with the inner sum replaced by K^\pm , hinges on Lemma 4.2 below and on Lemma 4.4, which will be proved in Step 4.

First, we need a representation of the Jacobi polynomials $P_d^{(\alpha, \beta)}$ showing explicitly the dependence on the parameters. To this end, we use a very precise representation of $P_d^{(\alpha, \beta)}$ due to A. Fitohui and M. M. Hamza [FiH]. We denote by J_ν the Bessel function of order ν .

Lemma 4.2. *Let $\alpha > -\frac{1}{2}$, $\beta > -1$, d a positive integer. Then*

$$\begin{aligned} (\sin \theta)^{\alpha+1/2} (\cos \theta)^{\beta+1/2} P_d^{(\alpha, \beta)}(\cos 2\theta) \\ = \frac{\Gamma(d + \alpha + 1)}{d!} \theta^{1/2} \left(\sum_{p=0}^m \theta^p Q_{2p}(\beta, \theta) \frac{J_{\alpha+p}(N\theta)}{N^{\alpha+p}} + \theta^{m+1} \mathcal{R}_{m, N}(\theta) \right), \end{aligned} \quad (4.6)$$

where $N := 2d + \alpha + \beta + 1$, the functions $Q_{2p}(\beta, \theta)$ are polynomials of degree $2p$ in β and analytic in $\theta \in [0, \pi/2)$, and

$$\mathcal{R}_{m, N} = O(N^{-(\alpha+m+3/2)}), \quad (4.7)$$

as $N \rightarrow +\infty$, uniformly in $\theta \in [0, \frac{\pi}{2} - \tilde{\varepsilon}]$, $\tilde{\varepsilon} > 0$ being arbitrary.

Proof. This is just a restatement of Theorem 4 in [FiH]; notice however that the parentheses are missing on the right hand side of (6.6) in [FiH]. By formula (6.6) in [FiH], setting $x = 2\theta$ and $B_p(x) = Q_{2p}(\beta, \theta)$ (and calling N what is $2N$ in [FiH]) we immediately obtain (4.6).

Next, the functions $B_p(x)$ are recursively defined by

$$B_0(x) = 1, \quad (x^{p+1} B_{p+1}(x))' = -\frac{1}{2} x^p \left(B_p''(x) + \frac{1-2\alpha}{x} B_p'(x) + \chi(x) B_p(x) \right), \quad (4.8)$$

where

$$\chi(x) = \left(\frac{1}{4} - \alpha^2 \right) \left(\frac{1}{4 \sin^2(x/2)} - \frac{1}{x^2} \right) + \left(\frac{1}{4} - \beta^2 \right) \frac{1}{4 \cos^2(x/2)},$$

and it turns out that the $B_p(x)$ are analytic for $x \in [0, \pi)$ (see Section 6.3 in [FiH]).

From the recursive relation (4.8) it is easy to see that the functions B_p are polynomials of degree $2p$ in the index β . (We point out that, for our purposes, the dependence on α is not relevant, since $\alpha = n - 1$ and it is not related to the indexes ℓ, ℓ' , while $\beta = |\ell' - \ell|$). In fact, the recursive relation can be restated by saying that

$$-2B_{p+1} = H_{p+1}((L + \chi)B_p),$$

where H_{p+1} is the integral operator (3.3) in [FiH], independent of α and β , and L is the differential operator given by $Lu = u'' + \frac{1-2\alpha}{x}u'$. Now, the statement about the dependence on β follows easily by induction. Hence, $Q_{2p}(\beta, \theta)$ is a polynomial of degree $2p$ in β and analytic in $\theta \in [0, \frac{\pi}{2})$.

The statement about the remainder term $\mathcal{R}_{m,N}(\theta)$ is explicit in Theorem 4 (see formula (6.6) again) in [FiH]. \square

We are going to apply Lemma 4.2, so that we observe that in our case $\alpha = n-1$, $d = \min\{\ell, \ell'\}$ and $\beta = |\ell' - \ell|$; hence $N = \ell + \ell' + n$.

In what follows we denote by $g_j, \tilde{g}_{j'}$ polynomials of degree j, j' resp., in the indicated variables, that again may have different expression from one line to the next. Then, we write

$$\frac{\ell + \ell' + n}{\ell'} \binom{\ell' + n - 1}{\ell' - 1} = (\ell + \ell' + n) \frac{(\ell' + n - 1)!}{\ell'!n!} = Ng_{n-1}(\ell'), \quad (4.9)$$

and

$$\frac{\Gamma(\ell + n)}{\ell!} = \tilde{g}_{n-1}(\ell). \quad (4.10)$$

Then, using (2.6), (4.9) and (4.10), writing $\langle z, w \rangle = e^{i\omega} \cos \theta$, in the case $\ell' \geq \ell$ we have

$$\begin{aligned} Z_{\ell, \ell'}(z, w) &= \frac{n}{\omega_{2n+1}} Ng_{n-1}(\ell') e^{i\omega(\ell' - \ell)} (\sin \theta)^{-n+1/2} (\cos \theta)^{-1/2} \tilde{g}_{n-1}(\ell) \theta^{1/2} \\ &\quad \times \left(\sum_{p=0}^m \theta^p Q_{2p}(\beta, \theta) \frac{J_{n-1+p}(N\theta)}{N^{n-1+p}} + \theta^{m+1} \mathcal{R}_{m,N}(\theta) \right) \\ &= b(\theta) e^{i\omega(\ell' - \ell)} Ng_{n-1}(\ell') \tilde{g}_{n-1}(\ell) \\ &\quad \times \left(\sum_{p=0}^m \theta^{2p} Q_{2p}(\beta, \theta) \frac{J_{n-1+p}(N\theta)}{(N\theta)^{n-1+p}} + \theta^{m-n+2} \mathcal{R}_{m,N}(\theta) \right), \end{aligned} \quad (4.11)$$

where b denotes an entire function of θ .

If $\ell \geq \ell'$ we simply switch the roles between ℓ and ℓ' in the formula above.

Then, in order to estimate (4.2) it suffices to bound the modulus of

$$\begin{aligned} &e^{i[t\lambda_{\ell, \ell'} + \omega(\ell' - \ell)]} \varphi(h^2 \lambda_{\ell, \ell'}) \psi(\ell'/\ell) \eta_{\pm}(\ell - \ell') Ng_{n-1}(\ell') \tilde{g}_{n-1}(\ell) \\ &\quad \times \left(\sum_{p=0}^m \theta^{2p} Q_{2p}(\beta, \theta) \frac{J_{n-1+p}(N\theta)}{(N\theta)^{n-1+p}} + \theta^{m-n+2} \mathcal{R}_{m,N}(\theta) \right). \end{aligned}$$

We need to distinguish the cases when $N\theta$ remains bounded and when it is bounded from below. Then, let χ_1 be a smooth cut-off function with compact support such that $0 \leq \chi_1 \leq 1$, $\chi_1(x) = 1$ for $0 \leq x \leq 1$ and $\chi_1(x) = 0$ for $x \geq 2$, and set $\chi_2 = 1 - \chi_1$. Therefore, for $j = 1, 2$ we write

$$\begin{aligned} K_j^{\pm}(\omega, \theta) &:= \sum_{a/h^2 < \lambda_{\ell, \ell'} < b/h^2} e^{i[t\lambda_{\ell, \ell'} + \omega(\ell' - \ell)]} \varphi(h^2 \lambda_{\ell, \ell'}) \psi(\ell'/\ell) \eta_{\pm}(\ell - \ell') Ng_{n-1}(\ell') \tilde{g}_{n-1}(\ell) \chi_j(N\theta) \\ &\quad \times \left(\sum_{p=0}^m \theta^{2p} Q_{2p}(\beta, \theta) \frac{J_{n-1+p}(N\theta)}{(N\theta)^{n-1+p}} + \theta^{m-n+2} \mathcal{R}_{m,N}(\theta) \right). \end{aligned} \quad (4.12)$$

Remark 4.3. We observe that the eigenvalue $\lambda_{\ell, \ell'} = 2[(\ell + n/2)(\ell' + n/2) - n^2/4]$ and we set

$$k = \ell + \frac{n}{2}, \quad k' = \ell' + \frac{n}{2}. \quad (4.13)$$

Notice that we may consider the quantities g_{n-1}, \tilde{g}_{n-1} as functions of k, k' resp., and write $N = k + k'$. We adopt the convention that, if n is odd, then the symbol $\sum_{k, k' \geq 1}$ shall denote the sum over a suitable subset of \mathbf{N} shifted by $1/2$.

Moreover, the condition $h^2 \lambda_{\ell, \ell'} \in \text{supp } \varphi$ becomes

$$\frac{a_h}{h^2} \leq kk' \leq \frac{b_h}{h^2},$$

where we set

$$a_h = \frac{a}{2} + \frac{h^2 n^2}{4}, \quad \text{and} \quad b_h = \frac{b}{2} + \frac{h^2 n^2}{4}.$$

For simplicity of notation we take $0 < a' \leq a_h$ and $b' \geq b_h$ for all $h \leq 1$. We also set $c' = \sqrt{b'}$.

The cut-off function $\varphi(h^2 \lambda_{\ell, \ell'})$ can be written as

$$\varphi(h^2 \lambda_{\ell, \ell'}) = \varphi(h^2(2kk' - n^2/2)) =: \varphi_h(h^2 kk').$$

We remark that Lemma 4.4 holds true if the cut-off function φ is replaced by a family of functions φ_ε converging in the Schwartz norms to φ as $\varepsilon \rightarrow 0$. Since the dependence on h of φ_h is ineffective, with an abuse of notation, we write again φ to denote the functions φ_h .

The cut-off function $\psi(\ell/\ell')$, supported when $1/M \leq \ell/\ell' \leq M$ is changed into

$$\psi((k - n/2)/(k' - n/2)) =: \tilde{\psi}(k, k').$$

Observed that $\tilde{\psi}$ is a cut-off function having support contained in the set $\{1 \leq k/k' \leq M\}$. Finally, notice that the support condition of φ implies that

$$n/2 \leq k, k' \leq c'/h.$$

With the change of parameters (4.13), the quantity β remains unchanged, and the phase function $t[\lambda_{\ell, \ell'} + \omega\beta]$ becomes $2t[kk' + \omega\beta - n^2/4]$ and we may absorb the factor 2 in the parameter t .

Step 4. We now wish to estimate the modulus of $K_1^\pm(\omega, \theta)$, as defined in (4.12).

We will consider the case of K_1^+ , the other one being completely analogous.

It suffices to estimate the modulus of

$$\begin{aligned} & \sum_{p=0}^m \left(\sum_{k, k' \geq n/2} e^{i[tkk' + \omega(k' - k)]} \varphi(h^2 kk') \tilde{\psi}(k, k') \chi_1(N\theta) \eta_+(k' - k) N g_{n-1}(k') \tilde{g}_{n-1}(k) \right. \\ & \quad \left. \times \theta^{2p} Q_{2p}(\beta, \theta) \frac{J_{n-1+p}(N\theta)}{(N\theta)^{n-1+p}} \right) \\ & + \sum_{k, k' \geq n/2} e^{i[tkk' + \omega(k' - k)]} \varphi(h^2 kk') \tilde{\psi}(k, k') \eta_+(k' - k) \chi_1(N\theta) N g_{n-1}(k') \tilde{g}_{n-1}(k) \theta^{m-n+2} \mathcal{R}_{m, N}(\theta) \\ & =: \sum_{p=0}^m K_p^{1,+}(\omega, \theta) + K_{\mathcal{R}}^{1,+}(\omega, \theta). \end{aligned} \tag{4.14}$$

Since $\mathcal{R}_{m, N}(\theta) = O\left(\frac{1}{N^{n+m+1/2}}\right)$ uniformly for $\theta \in [0, \frac{\pi}{2} - \varepsilon_1]$, for $\varepsilon_1 > 0$, we can easily estimate the modulus of the error term $K_{\mathcal{R}}^{1,+}(\omega, \theta, N)$ by simply taking the modulus inside the sum. Observing that $x \mapsto x^{n-1} \varphi(x)$ is also a smooth function with compact support, and choosing

$$m = \max(n - 2, 0)$$

$$\begin{aligned}
|K_{\mathcal{R}}^{1,+}(\omega, \theta)| &= \left| \sum_{k, k' \geq n/2} e^{i[tkk' + \omega(k' - k)]} \varphi(h^2 kk') \tilde{\psi}(k, k') \eta_+(k' - k) \chi_1(N\theta) \right. \\
&\quad \left. \times N g_{n-1}(k') \tilde{g}_{n-1}(k) \theta^{m-n+2} \mathcal{R}_{m,N}(\theta) \right| \\
&\leq \frac{C}{h^{2n-2}} \sum_{k=n/2}^{\lfloor c'/h \rfloor} \sum_{k'=n/2}^{\lfloor c'/h \rfloor} \left| \varphi(h^2 kk') (h^2 kk')^{n-1} \frac{1}{N^{n+m-1/2}} \right| \\
&\leq \frac{C}{h^{2n-2}} \sum_{k=n/2}^{\lfloor c'/h \rfloor} \sum_{k'=n/2}^{\lfloor c'/h \rfloor} \frac{1}{(k + k')^{n+m-1/2}} \\
&\leq \frac{C}{h^{2n-2}} \sum_{k=n/2}^{\lfloor c'/h \rfloor} \sum_{k'=n/2}^{\lfloor c'/h \rfloor} \frac{1}{(k + k')^{1/2}} \\
&\leq \frac{C}{h^{2n-1/2}}, \tag{4.15}
\end{aligned}$$

uniformly in ω and $\theta \in [0, \varepsilon_1]$. Therefore, with $m = \max(n - 2, 0)$ we have

$$|K_{\mathcal{R}}^{1,+}(\omega, \theta)| \leq \frac{C}{h^{2n-1/2}} \tag{4.16}$$

for all $n \geq 1$.

We turn to the estimate of the main term in (4.14), with $m = \max(n - 2, 0)$.

In order to take advantage of the oscillations of the kernel we need the following estimate for oscillating sums. We denote by $\hat{f}(\xi)$ the Fourier transform of an integrable function f and defined by $\hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-2\pi i x \xi} dx$.

Lemma 4.4. *Let $\varphi \in \mathcal{C}_0^\infty(\mathbf{R})$, $\text{supp } \varphi \subseteq [a, b]$, $0 < a < b < \infty$, and let σ be a symbol in \mathcal{S}^0 . Let $\mu \in \mathbf{R}$ and set $\text{dist}(\mu, \mathbf{Z}) \geq \delta$, for some $\delta > 0$. Then for every $L > 1$ there exists a positive constant $C_L > 0$, depending only on φ and σ , such that for $\delta, \varepsilon > 0$ with $0 < \varepsilon \leq \delta$, we have that*

$$\left| \sum_{k \in \mathbf{Z}} e^{2\pi i \mu k} \varphi(\varepsilon k) \sigma(k) \right| \leq C_L \max \left\{ \frac{\varepsilon^{L-1}}{\delta^L}, 1 \right\}, \tag{4.17}$$

as $\varepsilon \rightarrow 0$.

Proof. Let ψ, τ be such that $\varphi = \hat{\psi}$, $\sigma = \hat{\tau}$.

We consider first the case $\sigma = 1$. By the classical Poisson summation formula

$$\begin{aligned}
\left| \sum_{k \in \mathbf{Z}} e^{2\pi i \mu k} \varphi(\varepsilon k) \right| &= \left| \sum_{k \in \mathbf{Z}} e^{2\pi i \mu k} \widehat{\psi_\varepsilon}(k) \right| = \left| \sum_{k' \in \mathbf{Z}} \psi_\varepsilon(k' + \mu) \right| \\
&\leq \frac{C_L}{\varepsilon} \sum_{k' \in \mathbf{Z}} \frac{1}{\left(1 + \frac{|k' + \mu|}{\varepsilon}\right)^L}, \tag{4.18}
\end{aligned}$$

where $\psi_\varepsilon(x) = \varepsilon^{-1} \psi(\varepsilon^{-1} x)$.

We may assume $|\mu| \leq \frac{1}{2}$, so that $\text{dist}(\mu, \mathbf{Z}) = |\mu| \geq \delta$ and

$$\begin{aligned} \sum_{k' \in \mathbf{Z}} \frac{1}{(1 + \frac{|k' + \mu|}{\varepsilon})^L} &\leq \frac{1}{(1 + \frac{|\mu|}{\varepsilon})^L} + 2 \int_{-\infty}^{+\infty} \frac{1}{(1 + \frac{|x + \mu|}{\varepsilon})^L} dx \\ &\leq \frac{1}{(1 + \frac{\delta}{\varepsilon})^L} + 2\varepsilon \int_{-\infty}^{+\infty} \frac{1}{(1 + |y|)^L} dy. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \sum_{k \in \mathbf{Z}} e^{2\pi i \mu k} \varphi(\varepsilon k) \right| &\leq \frac{C_L}{\varepsilon} \left(\frac{1}{(1 + \frac{\delta}{\varepsilon})^L} + 2\varepsilon \int_{-\infty}^{+\infty} \frac{1}{(1 + |y|)^L} dy \right) \\ &\leq C_L \frac{\varepsilon^{L-1}}{(\varepsilon + \delta)^L} + C \\ &\leq C_L \max \left\{ \frac{\varepsilon^{L-1}}{\delta^L}, 1 \right\}, \end{aligned}$$

proving (4.17) in the case $\sigma = 1$.

Next we suppose that σ belongs to classical symbol class \mathcal{S}^0 . Notice that

$$\varphi(\varepsilon k) \sigma(k) = \widehat{\psi_\varepsilon}(k) \widehat{\tau}(k) = (\psi_\varepsilon * \tau)^\wedge(k) = ((\psi * \tau_{1/\varepsilon})_\varepsilon)^\wedge(k).$$

Thus we may repeat the previous arguments as in (4.18) to obtain

$$\begin{aligned} \left| \sum_{k \in \mathbf{Z}} e^{2\pi i \mu k} \varphi(\varepsilon k) \sigma(k) \right| &= \left| \sum_{k \in \mathbf{Z}} e^{2\pi i \mu k} ((\psi * \tau_{1/\varepsilon})_\varepsilon)^\wedge(k) \right| \\ &= \left| \sum_{k' \in \mathbf{Z}} (\psi * \tau_{1/\varepsilon})_\varepsilon(k' + \mu) \right| \\ &\leq \frac{C_L}{\varepsilon} \sum_{k' \in \mathbf{Z}} \frac{1}{(1 + \frac{|k' + \mu|}{\varepsilon})^L}, \end{aligned} \tag{4.19}$$

where C_L does not depend on ε as long as the Schwartz norms of $(\psi * \tau_{1/\varepsilon})$ are uniformly bounded in ε . This happens if and only if the Schwartz norms of $(\psi * \tau_{1/\varepsilon})^\wedge$ are uniformly bounded in ε , as $\varepsilon \rightarrow 0$. Now

$$(\psi * \tau_{1/\varepsilon})^\wedge(x) = \widehat{\psi}(x) \widehat{\tau_{1/\varepsilon}}(x) = \varphi(x) \sigma(x/\varepsilon).$$

Since $\sigma \in \mathcal{S}^0$, is straightforward to check that

$$\begin{aligned} \left| D_x^j (\varphi \sigma(\cdot/\varepsilon))(x) \right| &= \left| \sum_{j'=0}^j c_{j'} \varphi^{(j-j')}(x) \frac{1}{\varepsilon^{j'}} \sigma^{(j')}(x/\varepsilon) \right| \\ &\leq C \sum_{j'=0}^j |\varphi^{(j-j')}(x)| \frac{1}{\varepsilon^{j'}} \frac{1}{(1 + |x|/\varepsilon)^{j'}} \\ &\leq C \sum_{j'=0}^j |\varphi^{(j-j')}(x)| \frac{1}{(\varepsilon^{j'} + a)^{j'}}, \end{aligned}$$

since $\text{supp } \varphi \subset [a, b]$ and $a > 0$. The statement now follows. \square

Remark 4.5. It is worth noticing that, by choosing as a symbol σ a smooth cut-off function with compact support, the estimate (4.17) may be proved also for truncated sums.

Lemma 4.6. *For $N = k' + k$ and $\beta = |k' - k|$ set*

$$\sigma_1(k, k') = \eta_+(k' - k) \tilde{\psi}(k, k') \theta^{2p} Q_{2p}(\beta, \theta) \frac{J_{n-1+p}(N\theta)}{(N\theta)^{n-1+p}} \chi_1(N\theta).$$

Then σ_1 is a symbol of order 0 in k' , depending on the parameters θ and k , with norm uniformly bounded in such parameters.

Proof. We wish to show that, considering $k' = \xi$ as a continuous parameter, σ_1 is a smooth function of ξ and, for each non-negative integer k there exists a positive constant $C = C_k$, independent of k and $\theta \in [0, \pi/2 - \varepsilon_1]$, such that

$$|\partial_\xi^k \sigma_1(\xi)| \leq C(1 + |\xi|)^{-k}.$$

Notice that since $k' \geq 1$ we may assume that we have extended σ_1 to be identically 0 when $\xi \leq 1/2$.

Since the Bessel function J_ν of integral order ν is analytic and has a zero of order ν at the origin, it is clear that σ_1 is smooth and bounded uniformly in θ . Moreover, recall from Lemma 4.2 that $Q_{2p}(\beta, \theta)$ is a polynomial of degree $2p$ in β and analytic in $\theta \in [0, \frac{\pi}{2})$. Hence, since $\chi_1(N\theta) = 0$ for $N\theta \geq 2$, we have that, on the support of χ_1 , $\theta \leq 2/N \leq C/\xi$, implying that

$$|\theta^{2p} Q_{2p}(\beta, \theta)| \leq C \frac{|Q_{2p}(\beta, \theta)|}{N^{2p}}$$

which is bounded, as $\xi \rightarrow +\infty$, uniformly in θ .

Next we consider the derivatives. If the derivative falls on the factor $\theta^{2p} Q_{2p}(\beta, \theta)$ we simply lower the degree of the polynomial of ξ and then obtain the estimate

$$|\partial_\xi [\theta^{2p} Q_{2p}(\beta, \theta)]| \leq C \frac{1}{\xi},$$

as $\xi \rightarrow +\infty$, uniformly in θ , as we required.

If the derivative falls on the factor $\frac{J_{n-1+p}(N\theta)}{(N\theta)^{n-1+p}}$, since the derivative of the Bessel function of order ν satisfies the identity $J'_\nu(z) - \frac{\nu}{z} J_\nu(z) + J_{\nu-1}(z)$, again we obtain that

$$\left| \partial_\xi \left[\frac{J_{n-1+p}(N\theta)}{(N\theta)^{n-1+p}} \right] \right| \leq C \left(\theta + \frac{1}{N} \right) \leq C \frac{1}{\xi},$$

as $\xi \rightarrow +\infty$, uniformly in θ .

If the derivative falls on χ_1 , it produces an extra factor θ , which is less than C/ξ .

Hence,

$$|\partial_\xi \sigma_1(\xi)| \leq C(1 + |\xi|)^{-1},$$

as $|\xi| \rightarrow +\infty$, uniformly in θ .

Finally, if the derivative falls on $\tilde{\psi}$ it produces a factor of the order of k/ξ^2 which is less or equal to $C/|\xi|$, as $\xi \rightarrow +\infty$.

The argument can be repeated for all higher order derivatives, so the lemma is proven. \square

We wish to apply Lemma 4.4, and this leads us to analyze the phase function $tkk' + \omega(k' - k)$. Recall that A has been defined in (4.1) and that, as observed earlier, by passing to the complex conjugate, we may assume $t > 0$. We then introduce the set of indices in \mathbf{N}^2

$$\mathcal{V} = \{(k, k') : k, k' \geq n/2, a'/h^2 \leq kk' \leq b'/h^2, 1/M \leq k/k' \leq M, |k - k'| \geq 1/2\}.$$

We set moreover

$$\mathcal{V}_+ = \mathcal{V} \cap \{(k, k') : k' > k\} \quad \text{and} \quad \mathcal{V}_- = \mathcal{V} \cap \{(k, k') : k' < k\}. \quad (4.20)$$

Finally, we introduce the space of parameters (t, ω)

$$R := [h^2, h^s] \times [0, 2\pi) \subset A \times [0, 2\pi).$$

Lemma 4.7. *On \mathcal{V}_+ we set $\mu(k) := (tk + \omega)/2\pi$ and $\mu'(k') = (\omega - tk')/2\pi$. Then, there exist a constant $0 < \gamma < 1$ and two regions R_I, R_{II} in the (t, ω) -space, such that $R \subseteq \cup R_I$ and for all $(t, \omega) \in R_i$, $i \in \{I, II\}$, one between the following two conditions*

$$(I) \text{ dist}(\mu(k), \mathbf{Z}) \geq \gamma tk \text{ for all } (k, k') \in \mathcal{V}_+;$$

$$(II) \text{ dist}(\mu'(k'), \mathbf{Z}) \geq \gamma tk' \text{ for all } (k, k') \in \mathcal{V}_+$$

holds.

Analogous statement holds in the case of \mathcal{V}_- .

Proof. We begin by observing that $(k, k') \in \mathcal{V}_+$ implies that $1 \leq k < k' \leq \lfloor c'/h \rfloor$, so that, for $t \in [h^2, h^s]$ we have

$$\frac{h^2}{2\pi} \leq \frac{1}{2\pi} tk \leq \frac{1}{2\pi} tk' \leq ch^{s-1} \leq c_1/(4\pi), \quad (4.21)$$

if $h \leq \varepsilon_0$ is sufficiently small and for some positive, small enough c_1 .

Let $R_I = \{(t, \omega) \in [h^2, h^s] \times [0, 2\pi) : 0 < c_1 < \omega < 2\pi - c_1\}$, and c_1 as above. Then (I) holds for $(t, \omega) \in R_I$ since for m integer

$$\left| \frac{tk + \omega}{2\pi} - m \right| \geq \left| \frac{\omega}{2\pi} - m \right| - \frac{c_1}{4\pi} \geq \frac{c_1}{4\pi}.$$

Replacing ω by $2\pi - \omega$ we may assume now that $-c_1 < \omega < c_1$. In this case, notice that $\text{dist}(\mu, \mathbf{Z}) = \frac{1}{2\pi} |tk + \omega|$ and $\text{dist}(\mu', \mathbf{Z}) = \frac{1}{2\pi} |tk' - \omega|$.

Then, if $\omega > 0$, we have that $\frac{tk + \omega}{tk} \geq 1$, so that in this case (I) holds.

If $\omega < 0$, we then have that $\frac{tk' - \omega}{tk'} \geq 1$, so that in this case (II) holds. \square

Proposition 4.8. *There exists a constant $C > 0$ such that*

$$|K_p^{1,+}(\omega, \theta)| \leq C \frac{1}{|t|^{n+1}},$$

uniformly in ω and $\theta \in [0, \varepsilon_1]$, for all $|t| \in [h^2, ch^s]$.

Proof. Recall the definition of $K_p^{1,\pm}(\omega, \theta)$ introduced in (4.14).

Using Lemma 4.6 we have that

$$K_p^{1,+}(\omega, \theta) = \sum_{k, k' \geq 1} e^{i[tkk' + \omega|k' - k|]} \varphi(h^2 kk') N g_{n-1}(k') \tilde{g}_{n-1}(k) \sigma_1(k').$$

Notice that $g_{n-1}(k') \tilde{g}_{n-1}(k) = k^{n-1} k'^{n-1} +$ lower order terms. Thus, we estimate the higher order terms, the other ones being estimated in a similar way, giving rise to a better bound. Hence, it suffices to estimate

$$\left| \sum_{k, k' \geq n/2} e^{i[tkk' + \omega(k' - k)]} \varphi(h^2 kk') N k^{n-1} k'^{n-1} \sigma_1(k') \right|, \quad (4.22)$$

and we distinguish two cases according that condition (I) or (II) in Lemma 4.7 holds, respectively.

Case (I). We assume that $\text{dist}(\mu(k), \mathbf{Z}) \geq \gamma tk$ for all $(k, k') \in \mathcal{V}_+$ and for some $0 < \gamma < 1$, so that $|tk + \omega| \geq 2\pi\gamma tk$ for all $k \geq 1$. Starting from (4.22), we wish to estimate

$$\begin{aligned} & \left| \sum_{k=n/2}^{\lfloor c'/h \rfloor} \sum_{k'=n/2}^{\lfloor c'/h \rfloor} e^{i[tkk' + \omega(k'-k)]} \varphi(h^2 kk') N k^{n-1} k'^{n-1} \sigma_1(k, k') \right| \\ & \leq \frac{C}{h^{2n}} \sum_{k=n/2}^{\lfloor c'/h \rfloor} \frac{1}{k} \left| \sum_{k'=n/2}^{\lfloor c'/h \rfloor} e^{ik'[tk+\omega]} \varphi(h^2 kk') (h^2 kk')^n \sigma_1(k, k') \right| \\ & \quad + \frac{C}{h^{2n-2}} \sum_{k=n/2}^{\lfloor c'/h \rfloor} k \left| \sum_{k'=n/2}^{\lfloor c'/h \rfloor} e^{ik'[tk+\omega]} \varphi(h^2 kk') (h^2 kk')^{n-1} \sigma_1(k, k') \right|. \end{aligned} \quad (4.23)$$

Notice that it suffices to bound the first sum on the right hand side of (4.23) above.

We now apply Lemma 4.4 to the inner sum of the first term on the right hand side of (4.23) above, with

$$\mu = [tk + \omega]/2\pi, \quad \delta = \gamma tk \geq \gamma h^2 k = \varepsilon,$$

and cut-off function $x \mapsto x^n \varphi$ and with the aid of Lemma 4.6.

Hence we obtain that for every $L > 1$ the right hand side of (4.23) is less or equal to a constant times

$$\begin{aligned} & \frac{C_L}{h^{2n}} \sum_{k=n/2}^{\lfloor c'/h \rfloor} \frac{1}{k} \max \left(\frac{\varepsilon^{L-1}}{\delta^L}, 1 \right) \leq \frac{C}{h^{2n}} \sum_{k=n/2}^{\lfloor c'/h \rfloor} \frac{1}{k} \left(\frac{(h^2 k)^{(L-1)}}{(tk)^L} + 1 \right) \\ & \leq \frac{C}{h^{2n}} \left(\frac{h^{2(L-1)}}{t^L} \sum_{k=n/2}^{\lfloor c'/h \rfloor} \frac{1}{k^2} + \log(1/h) \right) \\ & \leq C \left(\frac{1}{t^{n+1}} + \frac{1}{h^{2(n+\kappa)}} \right), \end{aligned}$$

for any given $\kappa > 0$, if we choose $L = n + 1$.

Therefore, for t such that $h^2 \leq t \leq h^s$ for every $s > s_n$, we have

$$\frac{1}{h^{2(n+\kappa)}} \leq \frac{C}{t^{2(n+\kappa)/s}} \leq \frac{C}{t^{n+1}}.$$

Hence, for all $t \in A$ we obtain

$$|K_p^{1,+}(\omega, \theta)| \leq C \frac{1}{|t|^{n+1}}. \quad (4.24)$$

This proves the statement in Case (I).

Case (II). We now assume that $\text{dist}(\mu'(k'), \mathbf{Z}) \geq \gamma tk'$ for all $(k, k') \in \mathcal{V}_+$ and for some $0 < \gamma' < 1$, so that $|tk' - \omega| \geq 2\pi(h^2 k')^{\gamma'}$, all $k' \in \mathbf{N}$. In this case, again we start from (4.22), apply Lemma 4.4 to the inner sum with

$$\mu = [tk' - \omega]/2\pi, \quad \delta = \gamma tk' \geq \gamma h^2 k' = \varepsilon,$$

and cut-off function $x^n \varphi$ and with the aid of Lemma 4.6.

We have that

$$\begin{aligned}
& \left| \sum_{k, k' \geq n/2} e^{i[tkk' + \omega(k' - k)]} \varphi(h^2 kk') N k^{n-1} k'^{n-1} \sigma_1(k, k') \right| \\
& \leq \frac{C}{h^{2n}} \sum_{k' = n/2}^{\lfloor c'/h \rfloor} \frac{1}{k'} \left| \sum_{k = n/2}^{\lfloor c'/h \rfloor} e^{ik[tk' - \omega]} \varphi(h^2 kk') (h^2 kk')^n \sigma_1(k, k') \right| \\
& \quad + \frac{C}{h^{2n-2}} \sum_{k' = n/2}^{\lfloor c'/h \rfloor} k' \left| \sum_{k = n/2}^{\lfloor c'/h \rfloor} e^{ik[tk' - \omega]} \varphi(h^2 kk') (h^2 kk')^{n-1} \sigma_1(k, k') \right|. \tag{4.25}
\end{aligned}$$

As in the previous case we may limit ourselves to consider the first sum in (4.25), the estimate for the latter one being analogous. We have

$$\begin{aligned}
& \frac{1}{h^{2n}} \sum_{k' = 1}^{\lfloor c'/h \rfloor} \frac{1}{k'} \left| \sum_{k = 1}^{\lfloor c'/h \rfloor} e^{ik[tk' - \omega]} \varphi(h^2 kk') (h^2 kk')^n \sigma_1(k, k') \right| \\
& \leq \frac{C}{h^{2n}} \sum_{k' = 1}^{\lfloor c'/h \rfloor + 1} \frac{1}{k'} \left(\frac{(h^2 k')^{(L-1)}}{(tk')^L} + 1 \right) \\
& \leq \frac{C}{h^{2n}} \left(\frac{h^{2(L-1)}}{t^L} \sum_{k' = 1}^{\lfloor c'/h \rfloor} \frac{1}{k'^2} + \log(1/h) \right) \\
& \leq C \left(\frac{1}{t^{n+1}} + \frac{1}{h^{2(n+\kappa)}} \right),
\end{aligned}$$

choosing $L = n + 1$, for any $\kappa > 0$.

Arguing as in (4.24) in Case (I), for $h^2 \leq |t| \leq h^s$ we obtain

$$|K_p^{1,+}(\omega, \theta)| \leq C \frac{1}{|t|^{n+1}}. \tag{4.26}$$

The result now follows. \square

Step 5. Finally, we wish to estimate the modulus of $K_2^\pm(\omega, \theta)$, as defined in (4.12). Again, we consider only the case of K_2^+ .

In this case, it turns out that it suffices to take $m = 0$. We introduce the same spectral decomposition as in (4.3). Thus, we are led to consider the remainder term

$$\begin{aligned}
|K_{\mathcal{R}}^{2,+}(\omega, \theta)| &= \left| \sum_{k, k' \geq n/2} e^{i[tkk' + \omega(k' - k)]} \varphi(h^2 kk') \tilde{\psi}(k, k') \eta_+(k' - k) \chi_2(N\theta) \right. \\
& \quad \left. \times N g_{n-1}(k') \tilde{g}_{n-1}(k) \theta^{m-n+2} \mathcal{R}_{m,N}(\theta) \right| \\
&\leq \frac{C}{h^{2n-2}} \sum_{k = n/2}^{\lfloor c'/h \rfloor} \sum_{k' = n/2}^{\lfloor c'/h \rfloor} \left| \varphi(h^2 kk') (h^2 kk')^{n-1} \frac{\theta^{m-n+2}}{N^{n+m-1/2}} \right| \\
&\leq \frac{C}{h^{2n-2}} \sum_{k = n/2}^{\lfloor c'/h \rfloor} \sum_{k' = n/2}^{\lfloor c'/h \rfloor} \frac{\theta^{m+3/2}}{(N\theta)^{n-1/2} N^m},
\end{aligned}$$

so that, by choosing $m = 0$, we obtain

$$|K_{\mathcal{R}}^{2,+}(\omega, \theta)| \leq \frac{C}{h^{2n}} \quad (4.27)$$

for all $n \geq 1$, uniformly in ω and $\theta \in [0, \varepsilon_1]$.

Thus, for $j = 2$ we are led to consider the main term in (4.12), that is,

$$\begin{aligned} \Upsilon^+(\omega, \theta) := & \sum_{k, k' \geq n/2} e^{i[tkk' + \omega|k' - k|]} \varphi(h^2 k k') \tilde{\psi}(k, k') \eta_+(k' - k) \chi_2(N\theta) \\ & \times N g_{n-1}(k') \tilde{g}_{n-1}(k) \frac{J_{n-1}(N\theta)}{(N\theta)^{n-1}}. \end{aligned} \quad (4.28)$$

Recall that in this case we have that $N\theta \geq 1$. Then we use the asymptotic expansion of the Bessel function J_ν

$$J_\nu(x) = \frac{1}{x^{1/2}} \rho_1(x) e^{ix} + \frac{1}{x^{1/2}} \rho_2(x) e^{-ix} + \mathcal{O}(x^{-3/2}),$$

for some bounded functions ρ_j , and write $\Upsilon^+(\omega, \theta) = \Upsilon_1 + \Upsilon_2 + \Upsilon_3$, where, for $j = 1, 2$

$$\begin{aligned} \Upsilon_j(\omega, \theta) = & \sum_{k, k' \geq n/2} e^{i[tkk' + \omega(k' - k) \pm \theta(k + k')]} \varphi(h^2 k k') \tilde{\psi}(k, k') \eta_+(k' - k) \\ & \times N g_{n-1}(k') \tilde{g}_{n-1}(k) \frac{1}{(N\theta)^{n-1/2}} \rho_j(N\theta) \chi_2(N\theta) \end{aligned} \quad (4.29)$$

and

$$\begin{aligned} \Upsilon_3(\omega, \theta) = & \sum_{k, k' \geq 1} e^{i[tkk' + \omega(k' - k)]} \varphi(h^2 k k') \tilde{\psi}(k, k') \eta_+(k' - k) \\ & \times N g_{n-1}(k') \tilde{g}_{n-1}(k) \frac{1}{(N\theta)^{n-1}} \chi_2(N\theta) \mathcal{O}((N\theta)^{-3/2}). \end{aligned}$$

Lemma 4.9. *For $N = k' + k$, and $\beta = k' - k$, set*

$$\sigma_2(k, k') = \tilde{\psi}(k, k') \eta_+(k' - k) \frac{1}{(N\theta)^{n-\frac{1}{2}}} \chi_2(N\theta) R(N\theta),$$

where

$$R(N\theta) = \begin{cases} \rho_j(N\theta) & \text{for } \Upsilon_j, \ j = 1, 2, \\ (N\theta)^{1/2} \mathcal{O}((N\theta)^{-3/2}) & \text{for } \Upsilon_3. \end{cases} \quad (4.30)$$

Then σ_2 is a symbol of order 0 in k' (k resp.), depending on the parameters θ and k (k' resp.), with norm uniformly bounded in such parameters.

Proof. As in Lemma 4.6 we wish to show that, considering $k' = \xi$ as a continuous parameter, σ_2 is a smooth function of ξ and, for each non-negative integer m there exists a positive constant $C = C_m$, independent of k and $\theta \in (0, \varepsilon_1)$, such that

$$|\partial_\xi^m \sigma_2(\xi)| \leq C(1 + |\xi|)^{-m},$$

and we may assume that we have extended σ_2 to be identically 0 when $\xi \leq 1/2$.

Since $\chi_2(N\theta) = 0$ for $N\theta \leq 1$ it follows that σ_2 is smooth and bounded as $\xi \rightarrow +\infty$, uniformly in θ and k .

Next we consider the derivatives. If the derivative falls on the factor $\frac{1}{(N\theta)^{n-\frac{1}{2}}}$ (or on $\frac{1}{(N\theta)^{n-1}}$ in the case of $\Upsilon_{\mathcal{R}}$), using the condition $N\theta \geq 1$ we easily obtain, respectively, that

$$\left| \partial_\xi \left[\frac{1}{(N\theta)^{n-\frac{1}{2}}} \right] \chi_2(N\theta) \tilde{\psi}(k, k') \eta_+(k' - k) \right| \leq C \frac{1}{N} \leq C \frac{1}{\xi},$$

and

$$\left| \partial_\xi \left[\frac{1}{(N\theta)^{n-1}} \right] \chi_2(N\theta) \tilde{\psi}(k, k') \eta_+(k' - k) R(N\theta) \right| \leq C \frac{1}{N} \leq C \frac{1}{\xi},$$

as $\xi \rightarrow +\infty$, uniformly in θ .

If the derivative falls on $\rho_j(N\theta)$, by means of formula (15) in [St, p. 338] we observe that

$$\left| \partial_\xi \left[\sum_k a_{k,j} (\xi\theta)^{-k} \right] \tilde{\psi}(k, k') \eta_+(k' - k) \frac{1}{(N\theta)^{n-\frac{1}{2}}} \chi_2(N\theta) \right| \leq C \frac{1}{\xi},$$

where $a_{k,j}$, $j = 1, 2$, are suitable coefficients.

If the derivative falls on χ_2 , we notice that $\chi_2'(\xi) = 0$ unless $1 \leq \xi \leq 2$, so that

$$\left| \frac{1}{(N\theta)^{n-\frac{1}{2}}} \partial_\xi [\chi_2(N\theta)] \tilde{\psi}(k, k') \eta_+(k' - k) \right| \leq C\theta \leq C \frac{1}{N} \leq C \frac{1}{\xi},$$

and the same is true when $R(N\theta)$ is defined as in the latter case of (4.30).

Finally, if the derivative falls on the remainder term $\mathcal{O}((N\theta)^{-3/2})$ in the latter case of (4.30), we have

$$\left| \frac{1}{(N\theta)^{n-1}} \chi_2(N\theta) \tilde{\psi}(k, k') \eta_+(k' - k) \partial_\xi [\mathcal{O}((N\theta)^{-3/2})] \right| \leq C \frac{1}{N} \leq C \frac{1}{\xi}.$$

Hence,

$$|\partial_\xi \sigma_2(\xi)| \leq C(1 + |\xi|)^{-1},$$

as $|\xi| \rightarrow +\infty$, uniformly in θ .

The argument can be repeated for all higher order derivatives. In particular, when the derivatives involve the term $\mathcal{O}((N\theta)^{-3/2})$, we may use formula (8), p. 334 in [St], proving that this term behaves like a symbol of the expected order. Thus the lemma is proven. \square

In the case of Υ^+ we still need to use the oscillation of the phase and hence Lemma 4.4. Let R_I, R_{II} be as in Lemma 4.7. Since the phase in this case is $tkk' + \omega(k' - k) \pm \theta(k + k')$, and $\theta > 0$, we write $\tilde{\theta} = \pm\theta$ and let $|\tilde{\theta}|$ vary in $[1/N, \varepsilon_1]$. We then introduce the space of parameters (t, ω, θ)

$$R_\theta := \{(t, \omega, \tilde{\theta}) \in [h^2, h^s] \times [0, 2\pi) \times (-\varepsilon_1, \varepsilon_1) : |\tilde{\theta}| < tN/M_1 \text{ or } |\tilde{\theta}| > M_1 Nt\},$$

where $M_1 > 2(1 + M)$ is a large constant.

Lemma 4.10. *Let θ be such that $1 \leq N|\tilde{\theta}|$. Let \mathcal{V}_+ be defined as in (4.20). For $(k, k') \in \mathcal{V}_+$, set $\mu_2 = (tk + \omega + \tilde{\theta})/2\pi$ and $\mu'_2 = (tk' - \omega + \tilde{\theta})/2\pi$. Then, there exist a constant $\gamma > 0$ and finitely many regions in R_θ , such that for all $(t, \omega, \tilde{\theta})$ belonging to one of these regions, at least one between the following two conditions*

(III) $\text{dist}(\mu_2, \mathbf{Z}) \geq \gamma tk$ for all $(k, k') \in \mathcal{V}_+$,

(IV) $\text{dist}(\mu'_2, \mathbf{Z}) \geq \gamma tk'$ for all $(k, k') \in \mathcal{V}_+$

holds.

Proof. We begin observing that, if either $|\tilde{\theta}| < tN/M_1$ or $|\tilde{\theta}| > M_1 Nt$, then there exists a constant $C > 0$ such that

$$|tk + \tilde{\theta}| \geq Ctk \text{ and } |tk' + \tilde{\theta}| \geq Ctk'. \quad (4.31)$$

Then we split the proof in a few cases.

Case 1. Let $R_{\theta,I} := \{(t, \omega, \tilde{\theta}) \in R_{\theta} : 0 < c_1 < \omega < 2\pi - c_1\}$, where $c_1 > 10\varepsilon_1$ is a small constant. Then (III) holds for $(t, \omega, \tilde{\theta}) \in R_{\theta,I}$ since

$$\text{dist}(\mu_2, \mathbf{Z}) \geq \text{dist}(\omega/2\pi, \mathbf{Z}) - \frac{tk}{2\pi} - \frac{|\tilde{\theta}|}{2\pi} \geq \frac{c_1}{2\pi} - \frac{tk}{2\pi} - \frac{\varepsilon_1}{2\pi} \geq c_1 \left(\frac{1}{2\pi} - \frac{1}{20\pi} \right) - \frac{tk}{2\pi} \geq \gamma tk,$$

for all $k = 1, \dots, c'/h$, as a consequence of (4.21).

Hence, possibly replacing ω by $2\pi - \omega$, we may assume that $|\omega| \leq c_1$, and that $\text{dist}(\mu_2, \mathbf{Z}) = |tk + \omega + \tilde{\theta}|/2\pi$ and that $\text{dist}(\mu'_2, \mathbf{Z}) = |tk' - \omega + \tilde{\theta}|/2\pi$. Notice that now we may assume that $\tilde{\theta} < 0$ since otherwise (III) holds on R_I and (IV) on R_{II} , where R_I, R_{II} are defined in Lemma 4.7.

Case 2. Let $R_{\theta,II} := \{(t, \omega, \tilde{\theta}) \in R_{\theta} : |\omega| < c_2 t\}$, where $c_2 > 0$ is a small constant. If $(t, \omega, \tilde{\theta}) \in R_{\theta,II}$, then (III) holds for all $(k, k') \in \mathcal{V}_+$, since

$$|tk + \omega + \tilde{\theta}| \geq |tk + \tilde{\theta}| - |\omega| \geq t(Ck - c_2) \geq \gamma tk,$$

provided that c_2 is small enough.

Case 3. Next suppose $c_2 t \leq |\omega| \leq c_1$. If $\omega > 0$, then, if $tk + \tilde{\theta} > 0$, (III) holds. If $tk + \tilde{\theta} < 0$, we first observe that $|\tilde{\theta}| > M_1 tN$. Indeed, if $|\tilde{\theta}| < tN/M_1$, then

$$|\tilde{\theta}| < \frac{1}{M_1} t(k + k') \leq \frac{1}{M_1} tk(1 + M) \leq \frac{tk}{2},$$

since we chose $M_1 > 2(1 + M)$. Thus $tk + \tilde{\theta} > 0$, contradicting the hypothesis. Now it is easy to conclude that condition (IV) holds, since

$$tk' + \tilde{\theta} < tk' - M_1 tN < 0,$$

so that

$$|tk' + \tilde{\theta} - \omega| = \omega + |tk' + \tilde{\theta}| \geq Ctk'.$$

If $\omega < 0$, then, if $tk + \tilde{\theta} < 0$, (III) holds as a consequence of (4.31). If $tk + \tilde{\theta} > 0$, we notice that $|\tilde{\theta}| < \frac{1}{M_1} tN$. Then condition (IV) holds, since

$$tk' + \tilde{\theta} > tk' - \frac{1}{M_1} tN > tk'(1 - \frac{1}{M_1}) - \frac{M}{M_1} tk' > 0$$

provided that $M_1 > 1 + M$, so that $|tk' + \tilde{\theta} - \omega| = |\omega| + tk' + \tilde{\theta} \geq Ctk'$. \square

Proposition 4.11. *There exists a constant $C > 0$ such that, for all $|t| \in A$ when $n > 1$, and for all $h^2 \leq |t| \leq Ch^{4/3}$ when $n = 1$,*

$$|\Upsilon_j(\omega, \theta)| \leq C \frac{1}{|t|^{n+1}}$$

uniformly in ω and $\theta \in [0, \varepsilon_1]$, for $j = 1, 2$.

Proof. Recall that Υ_j have been defined in (4.29). Next, we fix a smooth cut-off function Ψ with compact support in $[1/M_1, M_1]$, where $M_1 > 1$ is a (large) constant. We decompose Υ_j by

setting

$$\begin{aligned}
& \Upsilon_j(\omega, \theta) \\
&= \sum_{k, k' \geq 1} e^{i[tkk' + \omega(k' - k) \pm \theta(k + k')]} \varphi(h^2 kk') \tilde{\psi}(k, k') \eta_+(k' - k) \chi_2(N\theta) \frac{Ng_{n-1}(k') \tilde{g}_{n-1}(k)}{(N\theta)^{n-1/2}} \\
&\quad \times \rho_j(N\theta) [1 - \Psi(\theta/tN)] \\
&\quad + \sum_{k, k' \geq 1} e^{i[tkk' + \omega(k' - k) \pm \theta(k + k')]} \varphi(h^2 kk') \tilde{\psi}(k, k') \eta_+(k' - k) \chi_2(N\theta) \frac{Ng_{n-1}(k') \tilde{g}_{n-1}(k)}{(N\theta)^{n-1/2}} \\
&\quad \times \rho_j(N\theta) \Psi(\theta/tN) \\
&=: \Upsilon_{j,1}(\omega, \theta) + \Upsilon_{j,2}(\omega, \theta), \tag{4.32}
\end{aligned}$$

for $j = 1, 2$. We also have

$$|\Upsilon_3(\omega, \theta)| = \left| \sum_{k, k' \geq 1} e^{i[tkk' + \omega(k' - k)]} \varphi(h^2 kk') N(kk')^{n-1} \sigma_2(k, k') \right|.$$

For $\Upsilon_{j,1}$ we argue as in the proof of Proposition 4.8 and divide into the cases in which (III) or (IV) hold, respectively. If, for instance, (III) holds, then as in the proof of Case (I) in Proposition 4.8, we have

$$\begin{aligned}
|\Upsilon_{j,1}(\omega, \theta)| &\leq \frac{C}{h^{2n-2}} \sum_{k=1}^{\lfloor c'/h \rfloor} \left| \sum_{k'=1}^{\lfloor c'/h \rfloor} e^{ik'[tk + \omega + \tilde{\theta}]} \varphi(h^2 kk') N(h^2 kk')^{n-1} \sigma_2(N) \right| \\
&\leq \frac{C}{h^{2n}} \sum_{k=1}^{\lfloor c'/h \rfloor} \frac{1}{k} \sum_{k'=\lfloor a'/(h^2 k) \rfloor}^{\lfloor b'/(h^2 k) \rfloor + 1} e^{ik'[tk + \omega \pm \theta]} \varphi(h^2 kk') (h^2 kk')^n \sigma_2(N) \Big| \\
&\quad + \frac{C}{h^{2n-2}} \sum_{k=1}^{\lfloor c'/h \rfloor} k \sum_{k'=\lfloor a'/(h^2 k) \rfloor}^{\lfloor b'/(h^2 k) \rfloor + 1} e^{ik'[tk + \omega \pm \theta]} \varphi(h^2 kk') (h^2 kk')^{n-1} \sigma_2(N) \Big| \\
&\leq \frac{C}{h^{2n+\kappa}},
\end{aligned}$$

for any $\kappa > 0$, uniformly in ω and $\theta \in [0, \varepsilon_1]$, by proceeding as in the proof of (4.24) by means of Lemma 4.10. The proof in the case in which condition (IV) holds is analogous to the proof of (4.26) in Proposition 4.8, and it is omitted.

Finally we estimate $\Upsilon_{j,2}$. In this sum we take advantage of the fact that θ/tN is bounded above and below from zero. We have

$$\begin{aligned}
|\Upsilon_{j,2}(\omega, \theta)| &\leq C \sum_{k, k' \geq 1} \varphi(h^2 k k') \tilde{\psi}(k, k') \eta_+(k' - k) \chi_2(N\theta) \frac{N^{2n-1}}{(N\theta)^{n-1/2}} \Psi(\theta/tN) \\
&\leq C \sum_{k, k' \geq 1} \varphi(h^2 k k') \tilde{\psi}(k, k') \eta_+(k' - k) \frac{N^{2n-1}}{(N^2 t)^{n-1/2}} \\
&\leq \frac{C}{t^{n-1/2}} \sum_{k, k' \geq 1} \varphi(h^2 k k') \tilde{\psi}(k, k') \\
&\leq \frac{C}{t^{n-1/2} h^2} \leq \frac{C}{t^{n+1+1/n-1/2}},
\end{aligned}$$

since $t \leq h^s$, so that $h^{-2} \leq t^{-2/s} < t^{-(n+1)/n}$. Thus when $n = 1$

$$|\Upsilon_{j,2}(\omega, \theta)| \leq \frac{C}{t^2}$$

for all $h^2 \leq |t| \leq ch^{4/3}$, while if $n > 1$

$$|\Upsilon_{j,2}(\omega, \theta)| \leq \frac{C}{t^{n+1}}$$

for all $|t| \in A$. Finally, we observe that Υ_3 may be treated as the sum in (4.22) by means of Lemma 4.7, so that

$$|\Upsilon_3(\omega, \theta)| \leq C \frac{1}{|t|^{n+1}},$$

uniformly in ω and $\theta \in [0, \varepsilon_1]$, for all $|t| \in [h^2, ch^s]$. \square

Thus, as a consequence of the decompositions (4.14) and (4.28), of Propositions 4.8 and 4.11, by using also (4.16) and (4.27), if $n > 1$ in (4.12) we obtain

$$|K_1^\pm(\omega, \theta) + K_2^\pm(\omega, \theta)| \leq \frac{C}{|t|^{n+1}},$$

uniformly in ω and $\theta \in [0, \pi/2 - \varepsilon_1]$, so that we finally get (4.2), that is,

$$\sup_{(z, w) \in \Omega} \left| \sum_{\ell, \ell'=1}^{+\infty} e^{it\lambda_{\ell, \ell'}} \varphi(h^2 \lambda_{\ell, \ell'}) \psi(\ell'/\ell) Z_{\ell, \ell'}(z, w) \right| \leq C \frac{1}{h^{2n+\kappa}},$$

for all $t \in A$, $n > 1$. When $n = 1$, as a consequence of Proposition 4.11 we get

$$\sup_{(z, w) \in \Omega} \left| \sum_{\ell, \ell'=1}^{+\infty} e^{it\lambda_{\ell, \ell'}} \varphi(h^2 \lambda_{\ell, \ell'}) \psi(\ell'/\ell) Z_{\ell, \ell'}(z, w) \right| \leq C \frac{1}{t^2}$$

for all $|t| \in [h^2, ch^{4/3}]$.

This concludes the proof of Theorem 3.1. \square

5. THE STRICHARTZ ESTIMATE

In this section we complete the proof of Theorem 1.1.

Following a classical pattern we invoke a result by Keel and Tao [KT]. Consider the family of operators

$$U(t) := \chi_J(t) e^{it\mathcal{L}} \varphi(h^2\mathcal{L}),$$

where $t \in \mathbf{R}$, $h \in (0, 1]$, χ_J denotes the characteristic function of the interval J and $|J| \approx h^\alpha$, where, if $n > 1$, we will select either $\alpha = s > s_n$ (with s_n given by (1.6)) or $\alpha = 2$. If $n = 1$, we will select either $\alpha \geq 4/3$ or $\alpha = 2$.

Then $U(t)$ satisfies the energy estimate $\|U(t)\|_{(L^2, L^2)} \leq C$ for some positive constant C and the following *untruncated decay estimate*

$$\begin{aligned} \|U(t)U(\tau)^* v_0\|_{L^\infty} &\| \chi_J(t - \tau) e^{i(t-\tau)\mathcal{L}} \varphi(h^2\mathcal{L}) v_0 \|_{L^\infty} \\ &\leq \frac{C}{|t - \tau|^{Q/2}} \|v_0\|_{L^1(S^{2n+1})} \end{aligned}$$

for all $t, \tau \in \mathbf{R}$, $t \neq \tau$. Hence Theorem 1.2 in [KT] yields the following result.

Proposition 5.1. *For any fixed $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}_+)$, there exists a constant $C > 0$ such that for all $h \in (0, 1]$, for any interval J of length $|J| \leq h^\alpha$ and for all $v_0 \in \mathcal{C}^\infty(S^{2n+1})$ the following estimate holds*

$$\left(\int_J \|e^{it\mathcal{L}} \varphi(h^2\mathcal{L}) v_0\|_{L^q}^p dt \right)^{1/p} \leq C \|v_0\|_{L^2} \quad (5.1)$$

for all pairs $(p, q) \neq (2, +\infty)$, satisfying (1.5), where $\alpha = 2$ if $v_0 \in \mathcal{C}_\mathcal{E}^\infty$ and $\alpha > s_n$ if $v_0 \in \mathcal{C}_\mathcal{V}^\infty$, and \mathcal{E}, \mathcal{V} are defined in (2.9) and (2.10). Here C depends only on p, q, n and s .

End of the proof of Theorem 1.1. Let $v_0 \in \mathcal{C}_\mathcal{V}^\infty$, $v_0 \in \mathcal{C}_\mathcal{E}^\infty$ respectively, and $\alpha = s > s_n$, $\alpha = 2$ respectively.

By writing $[-1, 1] = \cup_{k=1}^N J_k$, with J_k intervals, $|J_k| \approx h^\alpha$ and $N \approx h^{-\alpha}$, we have

$$\begin{aligned} \int_{-1}^1 \|e^{it\mathcal{L}} \varphi(h^2\mathcal{L}) v_0\|_{L^q}^p dt &\leq \sum_{k=1}^N \int_{J_k} \|e^{it\mathcal{L}} \varphi(h^2\mathcal{L}) v_0\|_{L^q}^p dt \\ &\leq CN \|v_0\|_{L^2}^p \\ &\leq Ch^{-\alpha} \|v_0\|_{L^2}^p, \end{aligned}$$

so that

$$\left(\int_{-1}^1 \|e^{it\mathcal{L}} \varphi(h^2\mathcal{L}) v_0\|_{L^q}^p dt \right)^{1/p} \leq Ch^{-\alpha/p} \|v_0\|_{L^2}. \quad (5.2)$$

Now, let $\tilde{\varphi} \in \mathcal{C}_0^\infty(\mathbf{R}_+)$ be such that $\tilde{\varphi}\varphi = \varphi$. Then (5.2), with φ replaced by $\tilde{\varphi}$ and the initial datum $\varphi(h^2\mathcal{L})v_0$, gives

$$\left(\int_{-1}^1 \|e^{it\mathcal{L}} \tilde{\varphi}(h^2\mathcal{L}) \varphi(h^2\mathcal{L}) v_0\|_{L^q}^p dt \right)^{1/p} \leq Ch^{-\alpha/p} \|\varphi(h^2\mathcal{L}) v_0\|_{L^2},$$

that is,

$$\left(\int_{-1}^1 \|e^{it\mathcal{L}} \varphi(h^2\mathcal{L}) v_0\|_{L^q}^p dt \right)^{1/p} \leq Ch^{-\alpha/p} \|\varphi(h^2\mathcal{L}) v_0\|_{L^2}. \quad (5.3)$$

We now apply Theorem 2.2 to $f = v(t) = e^{it\mathcal{L}}v_0$ and then we take the L^p -norm with respect to the variable t on $[-1, 1]$ and obtain that

$$\|v\|_{L^p([-1,1], L^q(S^{2n+1}))} \leq C \left(\|v_0\|_{L^2(S^{2n+1})} + \left\| \left(\sum_{j=1}^{+\infty} \|e^{it\mathcal{L}}\psi(2^{-2j}\mathcal{L})v_0\|_{L^q(S^{2n+1})}^2 \right)^{1/2} \right\|_{L^p([-1,1])} \right).$$

Next, let v_0 be any function in $\mathcal{C}^\infty(S^{2n+1})$. Since $p \geq 2$, by Minkowski's integral inequality we have

$$\begin{aligned} & \left\| \left(\sum_{j=1}^{+\infty} \|e^{it\mathcal{L}}\psi(2^{-2j}\mathcal{L})v_0\|_{L^q(S^{2n+1})}^2 \right)^{1/2} \right\|_{L^p([-1,1])} \\ & \leq \left(\sum_{j=1}^{+\infty} \left(\int_{-1}^1 \|e^{it\mathcal{L}}\psi(2^{-2j}\mathcal{L})v_0\|_{L^q(S^{2n+1})}^p dt \right)^{2/p} \right)^{1/2} \\ & \leq \left(\sum_{j=1}^{+\infty} \left(\int_{-1}^1 \|e^{it\mathcal{L}}\psi(2^{-2j}\mathcal{L})\pi_{\mathcal{V}}v_0\|_{L^q(S^{2n+1})}^p dt \right)^{2/p} \right)^{1/2} \\ & \quad + \left(\sum_{j=1}^{+\infty} \left(\int_{-1}^1 \|e^{it\mathcal{L}}\psi(2^{-2j}\mathcal{L})\pi_{\mathcal{E}}v_0\|_{L^q(S^{2n+1})}^p dt \right)^{2/p} \right)^{1/2} \\ & \leq C \left(\sum_{j=1}^{+\infty} 2^{(2js/p)} \|\psi(2^{-2j}\mathcal{L})\pi_{\mathcal{V}}v_0\|_{L^2(S^{2n+1})}^2 \right)^{1/2} + C \left(\sum_{j=1}^{+\infty} 2^{(4j/p)} \|\psi(2^{-2j}\mathcal{L})\pi_{\mathcal{E}}v_0\|_{L^2(S^{2n+1})}^2 \right)^{1/2} \\ & \leq C \|v_0\|_{\mathcal{X}^{(s/p, 2/p)}}, \end{aligned}$$

where we used, in particular, the Strichartz estimate (5.3) for the spectral truncations. This yields (1.7). Analogous arguments lead to

$$\left\| \left(\sum_{j=1}^{+\infty} \|e^{it\mathcal{L}}\psi(2^{-2j}\mathcal{L})v_0\|_{L^q(S^3)}^2 \right)^{1/2} \right\|_{L^p([-1,1])} \leq C \|v_0\|_{\mathcal{X}^{(s/p, 2/p)}}$$

for all $s \geq 4/3$, when $n = 1$. \square

The following result follows at once from the Strichartz estimates (1.7) and Minkowski inequality.

Corollary 5.2. *If p and q satisfy $\frac{2}{p} + \frac{Q}{q} = \frac{Q}{2}$, $p \geq 2$, $q < \infty$, then for all $T > 0$ and for all $s > s_n$, s_n defined by (1.6), if $n > 1$, or for all $s \geq 4/3$ if $n = 1$, there exists $C = C(p, T, s)$ such that for every $f \in L^1([-T, T], \mathcal{X}^{s/p, 2/p})$ we have*

$$\left\| \int_0^t e^{i(t-t')\mathcal{L}} f(t') dt' \right\|_{L^p([-T, T], L^q(S^{2n+1}))} \leq C \|f\|_{L^1([-T, T], \mathcal{X}^{(s/p, 2/p)})}. \quad (5.4)$$

6. NONLINEAR SCHRÖDINGER EQUATION ON S^{2n+1}

In this section, we discuss some local well-posedness results for the nonlinear Schrödinger equation (1.1), where $u : \mathbf{R} \times S^{2n+1} \rightarrow \mathbf{C}$, and F is a non linear polynomial of degree β , with $F(0) = 0$.

We shall need the following product rule for the W^r -norm.

Lemma 6.1. *If $1 \leq r \leq \infty$, for every $f, g \in W^r \cap L^\infty$ we have*

$$\|fg\|_{W^r} \leq C(\|f\|_{L^\infty}\|g\|_{W^r} + \|f\|_{W^r}\|g\|_{L^\infty}).$$

Proof. By means of the Littlewood–Paley decomposition, the proof is straightforward (see, for instance, [ZZh, Prop. 4.7]). \square

Proposition 6.2. *Let $r > \frac{Q}{2} - \frac{1}{\max(\beta-1, 2)}$. Then*

(1) *for every initial datum $u_0 \in W_V^r(S^{2n+1})$ there exist $T > 0$ and a unique solution u of (1.1),*

$$u \in \mathcal{C}([-T, T], W^r(S^{2n+1})) \cap L^p([-T, T], L^\infty(S^{2n+1}));$$

(2) *if u_0 belongs to $W_V^{r'}$ for some $r' > r$, then $u \in \mathcal{C}([-T, T], W^{r'}(S^{2n+1}))$.*

Proof. The scheme of the proof in the Riemannian framework is now classical (see, for example, [CazWe]); our proof is inspired, in particular, by Proposition 3.1 in [BuGT1].

Take $s > s_n$ if $n > 1$, $s \geq 4/3$ if $n = 1$. In the light of (2), we may assume $r < \frac{Q}{2}$. Then choose $p > \max(\beta - 1, 2)$ such that $r > \frac{Q}{2} - \frac{2-s}{p}$.

We denote by \mathcal{Y}_T the function space

$$\mathcal{Y}_T := \mathcal{C}([-T, T], W^r(S^{2n+1})) \cap L^p([-T, T], W^{\tilde{\sigma}, q}(S^{2n+1})),$$

where p, q satisfy (1.5) and $\tilde{\sigma} = r - \frac{s}{p}$. Observe that this implies $\tilde{\sigma} > \frac{Q}{q}$. Then \mathcal{Y}_T is a Banach space endowed with the norm

$$\|u\|_{\mathcal{Y}_T} := \max_{|t| \leq T} \|u(t)\|_{W^r} + \|(I + \mathcal{L})^{\tilde{\sigma}/2} u\|_{L^p([-T, T], L^q)}.$$

By the Duhamel formula, we have only to prove that the functional

$$\Phi(u)(t) = e^{it\mathcal{L}}u_0 - i \int_0^t e^{i(t-t')\mathcal{L}} F(u(t')) dt'$$

is a contraction on some ball of \mathcal{Y}_T , centered at the origin, for a sufficiently small $T > 0$. We have

$$\begin{aligned} \|\Phi(u)\|_{\mathcal{Y}_T} &\leq \max_{|t| \leq T} \|e^{it\mathcal{L}}u_0\|_{W^r} + \max_{|t| \leq T} \left\| \int_0^t e^{i(t-t')\mathcal{L}} F(u(t')) dt' \right\|_{W^r} \\ &\quad + \|(I + \mathcal{L})^{\tilde{\sigma}/2} e^{it\mathcal{L}}u_0\|_{L^p([-T, T], L^q)} + \left\| (I + \mathcal{L})^{\tilde{\sigma}/2} \int_0^t e^{i(t-t')\mathcal{L}} F(u(t')) dt' \right\|_{L^p([-T, T], L^q)} \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned}$$

Clearly,

$$A_1 = \max_{|t| \leq T} \|(I + \mathcal{L})^{r/2} e^{it\mathcal{L}}u_0\|_{L^2} = \|(I + \mathcal{L})^{r/2}u_0\|_{L^2} = \|u_0\|_{W^r}. \quad (6.1)$$

Next, by applying Hölder's inequality (with indices $\alpha = \frac{p}{\beta-1}$ and $\alpha' = \frac{p}{p-\beta+1}$) and Lemma 6.1 we obtain

$$\begin{aligned} A_2 &\leq \int_{-T}^T \|F(u(t'))\|_{W^r} dt' \leq C \int_{-T}^T \|u(t')\|_{L^\infty}^{\beta-1} \|u(t')\|_{W^r} dt' \\ &\leq C \left(\int_{-T}^T dt' \right)^{1/\alpha'} \left(\int_{-T}^T \|u(t')\|_{L^\infty}^{(\beta-1)\alpha} dt' \right)^{1/\alpha} \|u\|_{L^\infty([-T, T], W^r)} \\ &\leq CT^{1-\frac{\beta-1}{p}} \|u\|_{L^p([-T, T], L^\infty)}^{\beta-1} \|u\|_{L^\infty([-T, T], W^r)}. \end{aligned}$$

Now observe that $\mathcal{Y}_T \subset L^p([-T, T], L^\infty)$ as a consequence of [Fo2, Th. 4.17], so that

$$\|u\|_{L^p([-T, T], L^\infty)}^{\beta-1} \|u\|_{L^\infty([-T, T], W^r)} \leq C \|u\|_{\mathcal{Y}_T}^{\beta-1} \|u\|_{L^\infty([-T, T], W^r)} \leq C \|u\|_{\mathcal{Y}_T}^\beta,$$

and therefore

$$A_2 \leq CT^{1-\frac{\beta-1}{p}} \|u\|_{\mathcal{Y}_T}^\beta. \quad (6.2)$$

Next, since p and q satisfy the admissibility condition (1.5) and $r \geq s/p + \tilde{\sigma}$, by Theorem 1.1 we have

$$\begin{aligned} A_3 &= \left(\int_{-T}^T \|e^{it\mathcal{L}}(I + \mathcal{L})^{\tilde{\sigma}/2} u_0\|_{L^q}^p dt' \right)^{1/p} \leq C_T \|(I + \mathcal{L})^{\tilde{\sigma}/2} u_0\|_{W^{s/p}} \\ &\leq C_T \|u_0\|_{W^{s/p+\tilde{\sigma}}} \leq C_T \|u_0\|_{W^r}. \end{aligned} \quad (6.3)$$

Now we estimate A_4 . By means of Corollary 5.2 we obtain

$$\begin{aligned} A_4 &\leq C_T \int_{-T}^T \|(I + \mathcal{L})^{\tilde{\sigma}/2} F(u(t'))\|_{W^{s/p}} dt' \\ &\leq C_T \int_{-T}^T \|F(u(t'))\|_{W^{\tilde{\sigma}+s/p}} dt'. \end{aligned}$$

Thus, by reasoning as for A_2 , we obtain

$$A_4 \leq CT^{1-\frac{\beta-1}{p}} \|u\|_{\mathcal{Y}_T}^\beta. \quad (6.4)$$

Then, putting together (6.1), (6.2), (6.3) and (6.4), we finally obtain

$$\|\Phi(u)\|_{\mathcal{Y}_T} \leq C \left(\|u_0\|_{W^r(S^{2n+1})} + T^{1-\frac{\beta-1}{p}} (\|u\|_{\mathcal{Y}_T}^\beta + \|u\|_{\mathcal{Y}_T}) \right).$$

By using similar arguments, we also obtain for $u, v \in \mathcal{Y}_T$,

$$\begin{aligned} \|\Phi(u) - \Phi(v)\|_{\mathcal{Y}_T} &\leq CT^{1-\frac{\beta-1}{p}} \left(\|u - v\|_{\mathcal{Y}_T}^\beta + \|u - v\|_{\mathcal{Y}_T} \right) \\ &\leq CT^{1-\frac{\beta-1}{p}} \|u - v\|_{\mathcal{Y}_T} \left(1 + \|u\|_{\mathcal{Y}_T} + \|v\|_{\mathcal{Y}_T} \right)^{\beta-1}. \end{aligned}$$

Thus there exists some ball $B(0, R)$ in X_T , with radius $R > 0$, such that Φ maps $B(0, R)$ into $B(0, R)$ and Φ is a contraction on $B(0, R)$. This proves (1). We omit the straightforward proof of the uniqueness. Finally, (2) is a consequence of the following estimate

$$\|\Phi(u) - \Phi(v)\|_{L^\infty(W^{r,2})} \leq C_r T^{1-\frac{\beta-1}{p}} \|u - v\|_{L^\infty(W^r)} \left(1 + \|u\|_{X_T} + \|v\|_{X_T} \right)^{\beta-1}.$$

□

The same procedure, applied to initial data u_0 spectrally localized in \mathcal{E} , leads to the following result.

Proposition 6.3. *Let F be a non linear polynomial of degree β , with $F(0) = 0$. Take*

$$r > \frac{Q}{2}.$$

Then for every initial datum $u_0 \in W_{\mathcal{E}}^r(S^{2n+1})$ there exist $T > 0$ and a unique solution u of (1.1),

$$u \in \mathcal{C}([-T, T], W^r(S^{2n+1})) \cap L^p([-T, T], L^\infty(S^{2n+1})).$$

Proof. The proof is very similar to that of Proposition 6.2, with s replaced by 2. □

In the light of Theorem 1.1, this result is not surprising and confirms the dyscrasia illustrated in Proposition 2.1.

Moreover, we recall in passing that the analogous result in the Riemannian context, that is, the local well-posedness of the non linear Schrödinger equation in H^r , $r > d/2$, can be obtained by the classical energy method.

7. FINAL REMARKS

7.1. Discussion of optimality. Strichartz estimates proved in Theorem 1.1 are, in general, not sharp. To study optimality, we may use some sharp estimates for the joint spectral projections $\pi_{\ell, \ell'}$, proved by the first author in [Ca1, Ca2].

Proposition 7.1. ([Ca1, Ca2]) *Let $n \geq 2$ and let ℓ, ℓ' be non-negative integers. Then for $q \geq 2$ we have*

$$\|\pi_{\ell, \ell'}\|_{(L^2, L^q)} \leq C (\lambda_{\ell, \ell'})^{\alpha(1/q, n)} (\ell + \ell' + 1)^{\beta(1/q, n)},$$

where

$$\alpha(1/q, n) = \begin{cases} n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2} & \text{if } q \geq 2\frac{2n+1}{2n-1} \\ \frac{1}{2}(\frac{1}{q} - \frac{1}{2}) & \text{if } 2 \leq q \leq 2\frac{2n+1}{2n-1}. \end{cases}$$

and

$$\beta(1/q, n) = \begin{cases} \frac{1}{2} & \text{if } q \geq 2\frac{2n+1}{2n-1} \\ (n + \frac{1}{2})(\frac{1}{2} - \frac{1}{q}) & \text{if } 2 \leq q \leq 2\frac{2n+1}{2n-1}. \end{cases}$$

The estimates above are sharp. More precisely, for the given $q \geq 2$ and ℓ, ℓ' non-negative integers, there exist $C > 0$ and spherical harmonics $h_{\ell, \ell'} \in \mathcal{H}^{\ell, \ell'}$ such that

$$\|h_{\ell, \ell'}\|_{L^q(S^{2n+1})} \geq \frac{1}{C} (\lambda_{\ell, \ell'} + 1)^{\alpha(1/q, n)} (\ell + \ell' + 1)^{\beta(1/q, n)} \|h_{\ell, \ell'}\|_{L^2(S^{2n+1})}. \quad (7.1)$$

We shall now test the Strichartz estimates on an eigenfunction of the sublaplacian \mathcal{L} corresponding to the eigenvalue $N = \lambda_{\ell, \ell'}$. Then we consider the solution of the homogeneous Schrödinger equation $v(t, z) = e^{-it\lambda_{\ell, \ell'}} v_0$, with initial datum $v_0 = h_{\ell, \ell'}$, where $h_{\ell, \ell'}$ is a spherical harmonic in $\mathcal{H}^{\ell, \ell'}$, satisfying the conditions in (7.1)

Then we have

$$\begin{aligned} \|v\|_{L^p(I, L^q(S^{2n+1}))} &= \left(\int_I \left(\int_{S^{2n+1}} |e^{-it\lambda_{\ell, \ell'}} h_{\ell, \ell'}(z)|^q d\sigma(z) \right)^{p/q} dt \right)^{1/p} \\ &= \ell(I)^{1/p} \|h_{\ell, \ell'}\|_{L^q(S^{2n+1})} \\ &\geq \frac{1}{C} (\lambda_{\ell, \ell'} + 1)^{\alpha(1/q, n)} (\ell + \ell' + 1)^{\beta(1/q, n)} \|h_{\ell, \ell'}\|_{L^2(S^{2n+1})}. \end{aligned}$$

Now if $(\ell, \ell') \in \mathcal{V}$, for some fixed proper cone \mathcal{V} , defined as in (2.9), and if p, q satisfy the admissibility condition (1.5), then

$$\|v\|_{L^p(I, L^q(S^{2n+1}))} \geq \frac{1}{C} (\lambda_{\ell, \ell'} + 1)^{\alpha+\beta/2} \|h_{\ell, \ell'}\|_{L^2(S^{2n+1})} \approx \|v_0\|_{W^{\alpha+\beta/2}(S^{2n+1})}.$$

It is easy to check that $s/p > s_n/p > \alpha + \beta/2$, so that this estimate does not provide the sharp bound.

If $(\ell, \ell') \in \mathcal{E}$, where \mathcal{E} is defined as in (2.10), then

$$\begin{aligned} \|v\|_{L^p(I, L^q(S^{2n+1}))} &\geq \frac{1}{C}(\lambda_{\ell, \ell'} + 1)^{\alpha+\beta} \|h_{\ell, \ell'}\|_{L^2(S^{2n+1})} \\ &\geq \frac{1}{C}(\lambda_{\ell, \ell'} + 1)^{\frac{2n}{pQ}} \|h_{\ell, \ell'}\|_{L^2(S^{2n+1})} \approx \frac{1}{C} \|v_0\|_{W^{4n/pQ}(S^{2n+1})}. \end{aligned}$$

Now observe that

$$\frac{4n}{pQ} \frac{2}{p} \left(1 - \frac{1}{n+1}\right) = s_n, \quad \text{for } n > 1,$$

so that

$$\|v\|_{L^p(I, L^q(S^{2n+1}))} \geq \frac{1}{C} \|v_0\|_{W^{\frac{2}{p}(1-1/(n+1))}}$$

for all (p, q) satisfying (1.5). Anyway, in Theorem 1.1 we proved that, if $(\ell, \ell') \in \mathcal{E}$, then the critical index is $2/p$, instead of s_n . In other words, the index $s > \frac{2}{p} [1 - \frac{1}{n+1}]$ in Theorem 1.1 would be sharp, up to the loss of ε derivatives, if we were able to prove an estimate like (1.7) with the space $\mathcal{X}^{(s/p, 2/p)}(S^{2n+1})$ replaced by $W^{s/p}$.

7.2. Comparison with other subriemannian frameworks. As recalled in the Introduction, it has been proved in [BaGX] that no (global in time) dispersive estimate may hold for solutions of the Schrödinger equation on \mathbf{H}_n . Anyway, the situation seems to be less rigid on the reduced Heisenberg group \mathbf{h}_n , defined as $\mathbf{h}_n : \mathbf{C}^n \times \mathbf{T}$, with product

$$(z, e^{it})(w, e^{it'}) := (z + w, e^{i(t+t'+\Im m z \bar{w})}),$$

with $z, w \in \mathbf{C}^n$, $t, s \in \mathbf{R}$, due to the compactness of the center. We point out that there is an intimate connection between the reduced Heisenberg group and the unit complex sphere, since \mathbf{h}_n turns out to be a contraction of S^{2n+1} (see [CaCi] for more details about the construction of this contractive map). A detailed discussion of (local in time) dispersive estimates and of Strichartz estimates for solutions of the Schrödinger equation on \mathbf{h}_n requires some additional care and will be presented elsewhere.

7.3. Discussion of the admissibility conditions. Admissibility condition (1.5) has been directly inspired by the scale invariance condition (1.10) on a Riemannian compact manifold of dimension d (which in turn has been inherited by the euclidean space \mathbf{R}^d), with the dimension d replaced by the homogeneous dimension Q . Anyway, on the CR sphere the notion of dilation, which leads to (1.10) in the euclidean context, is not intrinsic. An interesting possibility could be investigating scaling conditions in the subriemannian framework of the reduced Heisenberg group, where dilations are well defined as $\lambda \circ (z, t) = (\lambda z_1, \dots, \lambda z_n, \lambda^2 t)$, and then importing them on S^{2n+1} .

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